On Truncation of irreducible representations of Chevalley groups

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Abstract We prove part of a higher rank analogue of the Mazur-Gouvea Conjecture. More precisely, let $\tilde{\mathbf{G}}$ be a connected, reductive $\mathbb{Q}$-split group and let Γ be an arithmetic subgroup of $\tilde{\mathbf{G}}$. We show that the dimension of the slope α subspace of the cohomology of Γ with values in an irreducible $\tilde{\mathbf{G}}$-module L is bounded independently of L. The proof is elementary making only use of general principles of the representation theory of algebraic groups; it is based on consideration of certain truncations of irreducible representations of Chevalley groups.

Introduction

(0.1) The first boundedness result for the dimension of the slope subspaces of the cohomology groups of arithmetic subgroups has been obtained by Hida: he showed that the dimension of the slope 0-subspace of the cohomology of an arithmetic subgroup Γ in $\mathrm{GL}_2(\mathbb{Z})$ with coefficients in the irreducible representation L_k of highest weight $k \in \mathbb{N}_0$ is bounded independently of k (cf. e.g. [Hi], chapter 7.2). Hida even showed that this dimension is constant as a function of k and later he generalized his results to higher rank. Following a suggestion by R. Taylor, Buzzard extended Hida's result to spaces of arbitrary slope (but still considering arithmetic subgroups of GL_2) (cf. [Bu]). A. Pande, following the method of Buzzard/Taylor, further extended Buzzard's result to GL_2 over imaginary quadratic fields (cf. [P]). We also mention the recent work by Harder in the slope 0 case (cf. [H]).

(0.2) In this article we prove boundedness of the dimension of the slope subspaces for arithmetic subgroups in general reductive groups. This generalizes the results of Buzzard and Pande in the GL_2 case. To describe our result in more detail, let $\tilde{\mathbf{G}}$ be a connected, $\mathbb{Q}$-split reductive group. We choose a $\mathbb{Q}$-split torus $\tilde{\mathbf{T}}$ in $\tilde{\mathbf{G}}$ and we denote by $L_{\tilde{\lambda}}$ the irreducible $\tilde{\mathbf{G}}$-module of highest weight $\tilde{\lambda} \in X(\tilde{\mathbf{T}})$. We fix a prime p and we let $\Gamma \leq \tilde{\mathbf{G}}(\mathbb{Z})$ be a subgroup. As in the above mentioned works, we assume that Γ satisfies a certain level condition at p (cf. section (4.2) for a precise definition). We define the slope α subspace of the group cohomology $H^i(\Gamma, L_{\tilde{\lambda}})$ with respect to a normalized Hecke operator at p. Our main result then is as follows (cf. section (6.5)).

Theorem. *Let $\beta \in \mathbb{Q}_{\geq 0}$ and select $i \in \mathbb{N}_0$. There is a constant $C = C_{i,\beta}$ such that*

$$\sum_{0 \leq \alpha \leq \beta} \dim H^i(\Gamma, L_{\tilde{\lambda}})^\alpha \leq C$$

for all integral and dominant weights $\tilde{\lambda}$.

(0.3) We comment on the proof of the above Theorem. The proof is based on consideration of certain truncations of irreducible representations. The truncation of $L_{\tilde{\lambda}}$ of length r is defined as a quotient $L_{\tilde{\lambda}}(\mathbb{Z})/L_{\tilde{\lambda}}(\mathbb{Z},r)$; here $L_{\tilde{\lambda}}(\mathbb{Z})$ is a $\mathbb{Z}$-form of $L_{\tilde{\lambda}}$ and $L_{\tilde{\lambda}}(\mathbb{Z},r) \leq L_{\tilde{\lambda}}(\mathbb{Z})$ is a $\mathbb{Z}$-submodule, which in particular contains all weight spaces whose weight has "relative height" strictly larger than r (cf. (1.3)). Thus, we truncate from $L_{\tilde{\lambda}}(\mathbb{Z})$ all "sufficiently high" weight spaces. We note that $L_{\tilde{\lambda}}(\mathbb{Z})/L_{\tilde{\lambda}}(\mathbb{Z},r)$ still is a Γ-module. Truncations of irreducible GL_2-modules already appear in the above mentioned works of Buzzard and Pande as certain subspaces of symmetric powers of the standard representation of GL_2. Our definition is an extension to the higher rank case using the semisimple representation theory of algebraic Chevalley groups. Using some cohomological formalism (cf. (6.4) Proposition; this extends an argument of Hida; cf. [Hi], chapter 7.2) and the Theorem on elementary divisors (cf. (6.2.2) Proposition) we bound the dimension of the slope α subspace of $H^i(\Gamma, L_{\tilde{\lambda}})$ by the size of the finite group $H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z})/L_{\tilde{\lambda}}(\mathbb{Z},r))$, where r is the smallest integer bigger than α. Our Theorem then is an immediate consequence of the following

Proposition. (cf. (4.4.2) Proposition for a precise statement). *If $\tilde{\lambda}$ and $\tilde{\lambda}'$ are sufficiently large and congruent to each other modulo p^{2r} then the corresponding truncations of length r are isomorphic as Γ-modules, i.e. the isomorphism class of $L_{\tilde{\lambda}}(\mathbb{Z})/L_{\tilde{\lambda}}(\mathbb{Z},r)$ only depends on $\tilde{\lambda}$ modulo p^{2r}.*

We deduce the Proposition from its differential version for representations of semi simple Lie algebras by using the method of Chevalley groups. We note that in the slope 0 case (i.e. in the cases considered by Hida), the relevant truncations turn out to be isomorphic to $\mathbb{Z}/(p)$ and the Proposition holds trivially. Also, in the context of symmetric powers of the standard representation of GL_2 a result analogous to the above Proposition appears in [P].

(0.4) We add two Remarks. 1.) The assumption that $\tilde{\mathbf{G}}$ is $\mathbb{Q}$-split is necessary to define the Hecke operator. If we assume weaker that $\tilde{\mathbf{G}}$ is $\mathbb{Q}_p$-split then a version of the above Theorem with p-adic coefficients still holds (cf. section (6.6)) 2.) Buzzard and Pande make use of the fact that in the GL_2 case the (interesting) cohomology appears in degree 1, hence, any cocycle already is determined by its values on a set of generators of Γ. This does not hold in the higher rank case, where the (interesting) cohomology appears around the middle degree. On the other hand, Buzzard and Pande determine an explicit upper bound for the dimension of the slope spaces and in the case of the trivial congruence subgroup in a indefinite quaternion algebra Pande also proves local constancy of the dimension of the slope spaces. We hope to be able to deal with these questions in the context of this article in the future.

(0.5) Our motivation for writing this article comes from the Mazur-Gouvea Conjecture. On the one hand, the above Theorem confirms a part of a higher rank analogue of the Mazur-Gouvea Conjecture. On the other hand, we hope that it will be an ingredient in an approach to the full Mazur-Gouvea Conjecture (existence of p-adic families of modular forms), which is based on a rather elementary comparison of trace formulas and does not make use of advanced theories like rigid analytic geometry (cf. [Ma 1,2]).

1 Truncation of irreducible representations of semi simple Lie algebras

(1.1) Notations. We fix a complex semi-simple Lie algebra $\mathfrak{g}$ of rank ℓ. We introduce the following notations.

- $\mathfrak{h}$ denotes a fixed a Cartan subalgebra of $\mathfrak{g}$ and Φ is the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$.
- Δ is a fixed choice of a basis of Φ and Φ^+ resp. Φ^- is the set of positive resp. negative roots in Φ.
- $\mathfrak{g}(\alpha)$ is the weight α subspace of $\mathfrak{g}$ (with respect to the adjoint action of $\mathfrak{h}$) and we set $\mathfrak{n} = \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}(\alpha)$, $\mathfrak{n}^- = \bigoplus_{\alpha\in\Phi^-} \mathfrak{g}(\alpha)$.
- $(\quad,\quad)$ is the Killing form on $\mathfrak{g}$ and for $\lambda \in \mathfrak{h}^*$ we define $t_\lambda \in \mathfrak{h}$ by $(t_\lambda, h) = \lambda(h)$ for all $h \in \mathfrak{h}$. In this way, $(\quad,\quad)$ induces a pairing on $\mathfrak{h}^*$ and for $\lambda, \mu \in \mathfrak{h}^*$ we set $\langle\lambda, \mu\rangle = 2(\lambda, \mu)/(\mu, \mu)$.
- for every $\alpha \in \Phi$ we define the coroot $\alpha^\vee = h_\alpha = \frac{2t_\alpha}{(t_\alpha,t_\alpha)} \in \mathfrak{h}$; hence, $\lambda(h_\alpha) = \langle\lambda, \alpha\rangle$.
- we denote by Γ_{sc} the weight lattice of $\mathfrak{g}$ consisting of all integral weights in $\mathfrak{h}^*$, i.e. Γ_{sc} consists of all weights $\lambda \in \mathfrak{h}^*$ such that $\lambda(h_\alpha) \in \mathbb{Z}$ for all $\alpha \in \Phi$. Γ_{ad} is the root lattice, i.e. the subgroup of $\mathfrak{h}^*$ generated by the roots. We note that $\Gamma_{\mathrm{sc}} \leq \mathfrak{h}^*$ is a $\mathbb{Z}$-lattice in $\mathfrak{h}^*$ and Γ_{ad} is a sublattice with basis Δ. We write $\lambda \geq \mu$ if $\lambda - \mu$ is a linear combination of elements in Φ^+ with positive coefficients. We denote by

$$\mathrm{ht} = \mathrm{ht}_\Delta : \Gamma_{\mathrm{ad}} \to \mathbb{Z}$$

  the height function on Γ_{ad}, i.e. $\mathrm{ht}(\lambda) = \sum_{\alpha\in\Delta} c_\alpha$ for $\lambda = \sum_{\alpha\in\Delta} c_\alpha\alpha \in \Gamma_{\mathrm{ad}}$.
- $\mathcal{U}$ is the universal envelopping algebra of $\mathfrak{g}$ and $\mathcal{U}^+$, resp. $\mathcal{U}^-$ resp. $\mathcal{U}^o$ is the universal envelopping algebra of $\mathfrak{n}$ resp. $\mathfrak{n}^-$ resp. $\mathfrak{h}$.
- For any $\alpha \in \Phi$ we choose a root vector $x_\alpha \in \mathfrak{g}(\alpha)$ such that $\{x_\alpha,\ \alpha \in \Phi,\ h_\alpha,\ \alpha \in \Delta\}$ is a *Chevalley basis* of $\mathfrak{g}$.
- We fix an ordering $\{\alpha_1, \ldots, \alpha_s\}$ of Φ^+; for any multi index $\mathbf{n} = (n_1, \ldots, n_s) \in \mathbb{N}_0^s$ we set

$$X_+^{\mathbf{n}} = \frac{x_{\alpha_1}^{n_1}}{n_1!} \cdots \frac{x_{\alpha_s}^{n_s}}{n_s!} \in \mathcal{U}^+$$

  and

$$X_-^{\mathbf{n}} = \frac{x_{-\alpha_1}^{n_1}}{n_1!} \cdots \frac{x_{-\alpha_s}^{n_s}}{n_s!} \in \mathcal{U}^-.$$

  Moreover, for $\mathbf{n} = (n_\alpha)_{\alpha\in\Delta} \in \mathbb{N}_0^\Delta$ we set

$$H^{\mathbf{n}} = \prod_{\alpha\in\Delta} \binom{h_\alpha}{n_\alpha} \in \mathcal{U}^o,$$

  where $\binom{h}{n} = h(h-1)\cdots(h-n+1)/n!$.

- We set
$$\mathcal{U}_{\mathbb{Z}} = \bigoplus_{\mathbf{n}_1 \in \mathbb{N}_0^s, \mathbf{n}_2 \in \mathbb{N}_0^{\Delta}, \mathbf{n}_3 \in \mathbb{N}_0^s} \mathbb{Z}\, X_-^{\mathbf{n}_1} H^{\mathbf{n}_2} X_+^{\mathbf{n}_3}.$$
$\mathcal{U}_{\mathbb{Z}}$ is a $\mathbb{Z}$-algebra, hence, a $\mathbb{Z}$-form of the associative algebra $\mathcal{U}$. Similarly, $\mathcal{U}^+$ resp. $\mathcal{U}^-$ resp. $\mathcal{U}^o$ have $\mathbb{Z}$-forms $\mathcal{U}_{\mathbb{Z}}^+$ resp. $\mathcal{U}_{\mathbb{Z}}^-$ resp. $\mathcal{U}_{\mathbb{Z}}^o$ with $\mathbb{Z}$-basis $\{X_+^{\mathbf{n}},\ \mathbf{n} \in \mathbb{N}_0^s\}$ resp. $\{X_-^{\mathbf{n}},\ \mathbf{n} \in \mathbb{N}_0^s\}$ resp. $\{H^{\mathbf{n}},\ \mathbf{n} \in \mathbb{N}_0^{\Delta}\}$. In particular, we obtain $\mathcal{U}_{\mathbb{Z}} = \mathcal{U}_{\mathbb{Z}}^- \mathcal{U}_{\mathbb{Z}}^o \mathcal{U}_{\mathbb{Z}}^+$.

We denote by $[x]$ the largest integer, which is equal to or smaller than x. We denote by v_p the p-adic valuation on $\bar{\mathbb{Q}}_p$ normalized by $v_p(p) = 1$. Moreover, we fix an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$, hence, v_p defines a p-adic valuation on $\bar{\mathbb{Q}}$.

(1.2) Irreducible representations of semi simple Lie algebras. Let $\lambda \in \Gamma_{\text{sc}}$, i.e. λ is an integral and dominant weight. We denote by $(\rho_\lambda, L_\lambda)$ the finite dimensional, irreducible complex representation of $\mathfrak{g}$ of highest weight λ. We note that L_λ is an $\mathcal{U}$-module. We denote by $L_\lambda(\mu)$ the weight μ subspace of L_λ, Π_λ is the set of all nontrivial weights μ of L_λ (i.e $L_\lambda(\mu) \neq 0$) and $\Gamma_\lambda \leq \mathfrak{h}^*$ is the weight lattice of ρ_λ, i.e. Γ_λ is the subgroup of Γ_{sc} generated by Π_λ. Thus,
$$\Gamma_{\text{sc}} \geq \Gamma_\lambda \geq \Gamma_{\text{ad}}.$$
We fix a maximal vector $v_\lambda \in L_\lambda$, i.e. v_λ has weight λ and $L_\lambda = \mathcal{U}^- v_\lambda$, and we set $L_\lambda(\mathbb{Z}) = \mathcal{U}_{\mathbb{Z}}^- v_\lambda$. Theorem 27.1 b.) in [Hu], p. 158 and its proof imply that $L_\lambda(\mathbb{Z})$ is an admissible, i.e. $\mathcal{U}_{\mathbb{Z}}$-invariant lattice in L_λ; moreover, Theorem 27.1 a.) in [Hu], p. 158 implies
$$L_\lambda(\mathbb{Z}) = \bigoplus_{\mu \in \Pi_\lambda} L_\lambda(\mathbb{Z}, \mu),$$
where $L_\lambda(\mathbb{Z}, \mu) = L_\lambda(\mu) \cap L_\lambda(\mathbb{Z})$ is the weight μ subspace in $L_\lambda(\mathbb{Z})$. For any $\mathbb{Z}$-algebra R we set $L_\lambda(R) = L_\lambda(\mathbb{Z}) \otimes_{\mathbb{Z}} R$ and $L_\lambda(R, \mu) = L_\lambda(\mathbb{Z}, \mu) \otimes_{\mathbb{Z}} R$. Hence,
$$L_\lambda(R) = \bigoplus_{\mu \in \Pi_\lambda} L_\lambda(R, \mu).$$

(1.3) The truncating submodule of an irreducible representation. In the remainder of section 1 we fix a prime $p \in \mathbb{N}$ and we define the following ("Iwahori-type") $\mathbb{Z}$-subalgebra of $\mathcal{U}_{\mathbb{Z}}$:
$$\mathcal{S} = \mathbb{Z}[\frac{x_{-\alpha}^n}{n!},\ \alpha \in \Phi^+, n \in \mathbb{N}_0,\ p^{m\text{ht}(\alpha)} \frac{x_\alpha^m}{m!},\ \alpha \in \Phi^+, m \in \mathbb{N}_0] \leq \mathcal{U}_{\mathbb{Z}},$$
i.e. $\mathcal{S}$ is the $\mathbb{Z}$-subalgebra of $\mathcal{U}_{\mathbb{Z}}$ generated by the elements $\frac{x_{-\alpha}^n}{n!}$, $\alpha \in \Phi^+$, $n \in \mathbb{N}_0$ and $p^{m\text{ht}(\alpha)} \frac{x_\alpha^m}{m!}$, $\alpha \in \Phi^+$, $m \in \mathbb{N}_0$. We select an integral and dominant weight $\lambda \in \Gamma_{\text{sc}}$. $L_\lambda(\mathbb{Z})$ then becomes an $\mathcal{S}$-module via restriction. In the following we want to introduce the truncation of the $\mathcal{S}$-module $L_\lambda(\mathbb{Z})$. We define a (relative) height function
$$\text{ht}_\lambda : \Pi_\lambda \to \mathbb{N}_0$$

on the set of weights of L_λ by

$$\mathrm{ht}_\lambda(\mu) = \mathrm{ht}(\lambda - \mu)$$

(note that $\lambda - \mu \in \Gamma_{\mathrm{ad}}$, hence, $\mathrm{ht}(\lambda - \mu)$ is defined). Explicitly, if $\mu = \lambda - \sum_{\alpha\in\Delta} c_\alpha \alpha$, $c_\alpha \in \mathbb{N}_0$, then $\mathrm{ht}_\lambda(\mu) = \sum_{\alpha\in\Delta} c_\alpha$. We also note that

$$\mathrm{ht}_\lambda(\mu + \omega) = \mathrm{ht}_\lambda(\mu) - \mathrm{ht}(\omega) \tag{1}$$

for all $\mu \in \Pi_\lambda$ and all $\omega \in \Gamma_{\mathrm{ad}}$ such that $\mu + \omega$ again is contained in Π_λ. For any integer $r \in \mathbb{N}_0$ we then define the following $\mathbb{Z}$-submodule of the highest weight module $L_\lambda(\mathbb{Z})$:

$$L_\lambda(\mathbb{Z}, r) = \bigoplus_{\substack{\mu\in\Pi_\lambda \\ 0\le \mathrm{ht}_\lambda(\mu)\le r}} p^{r-\mathrm{ht}_\lambda(\mu)}\, L_\lambda(\mathbb{Z}, \mu) \oplus \bigoplus_{\substack{\mu\in\Pi_\lambda \\ \mathrm{ht}_\lambda(\mu)> r}} L_\lambda(\mathbb{Z}, \mu).$$

Again, we put $L_\lambda(R, r) = L_\lambda(\mathbb{Z}, r) \otimes_{\mathbb{Z}} R$ for any $\mathbb{Z}$-algebra R, hence,

$$L_\lambda(R, r) = \bigoplus_{\substack{\mu\in\Pi_\lambda \\ 0\le \mathrm{ht}_\lambda(\mu)\le r}} p^{r-\mathrm{ht}_\lambda(\mu)}\, L_\lambda(R, \mu) \oplus \bigoplus_{\substack{\mu\in\Pi_\lambda \\ \mathrm{ht}_\lambda(\mu)> r}} L_\lambda(R, \mu).$$

We note that $L_\lambda(\mathbb{Z}, r)$ is a $\mathbb{Z}$-lattice in $L_\lambda(\mathbb{Q}, r)$.

(1.4) Lemma. *$L_\lambda(\mathbb{Z}, r)$ is an $\mathcal{S}$-invariant submodule of $L_\lambda(\mathbb{Z})$.*

Proof. Since $\frac{x_{-\alpha}^n}{n!} L_\lambda(\mathbb{Z}) \subseteq L_\lambda(\mathbb{Z})$ we obtain for all $\mu \in \Pi_\lambda$ and $\alpha \in \Phi^+$

$$\frac{x_{-\alpha}^n}{n!}\, L_\lambda(\mathbb{Z}, \mu) \subseteq L_\lambda(\mu - n\alpha) \cap L_\lambda(\mathbb{Z}) = L_\lambda(\mathbb{Z}, \mu - n\alpha).$$

Since $\mathrm{ht}_\lambda(\mu - n\alpha) \ge \mathrm{ht}_\lambda(\mu)$ (cf. equation (1)) the definition of $L_\lambda(\mathbb{Z}, r)$ immediately implies that $\frac{x_{-\alpha}^n}{n!}\, L_\lambda(\mathbb{Z}, r) \subseteq L_\lambda(\mathbb{Z}, r)$.

To show that all generators $p^{n\mathrm{ht}(\alpha)} \frac{x_\alpha^n}{n!}$ with $\alpha \in \Phi^+$ leave $L_\lambda(\mathbb{Z}, r)$ invariant, we distinguish cases.
1. First, we consider weights $\mu \in \Pi_\lambda$ satisfying $\mathrm{ht}_\lambda(\mu) \le r$. We let $v \in p^{r-\mathrm{ht}_\lambda(\mu)}\, L_\lambda(\mathbb{Z}, \mu)$ be arbitrary. For all $\alpha \in \Phi^+$ we obtain using equation (1)

$$p^{n\mathrm{ht}(\alpha)} \frac{x_\alpha^n}{n!}(v) \in p^{n\mathrm{ht}(\alpha)} p^{r-\mathrm{ht}_\lambda(\mu)} L_\lambda(\mathbb{Z}, \mu + n\alpha) = p^{r-\mathrm{ht}_\lambda(\mu+n\alpha)}\, L_\lambda(\mathbb{Z}, \mu + n\alpha).$$

Since $\mathrm{ht}_\lambda(\mu + n\alpha) \le \mathrm{ht}_\lambda(\mu) \le r$ we deduce that $p^{r-\mathrm{ht}_\lambda(\mu+n\alpha)} L_\lambda(\mathbb{Z}, \mu + n\alpha)$ is contained in $L_\lambda(\mathbb{Z}, r)$. Since v was arbitrary this implies that $p^{n\mathrm{ht}(\alpha)} \frac{x_\alpha^n}{n!} p^{r-\mathrm{ht}_\lambda(\mu)} L_\lambda(\mathbb{Z}, \mu) \subseteq L_\lambda(\mathbb{Z}, r)$ (note that we may assume that $\mu + n\alpha \in \Pi_\lambda$ since otherwise $x_\alpha^n v = 0$).
2. Second, we consider weights $\mu \in \Pi_\lambda$ with $\mathrm{ht}_\lambda(\mu) > r$. We let $v \in L_\lambda(\mathbb{Z}, \mu)$ be arbitrary. As above we find

$$p^{n\mathrm{ht}(\alpha)} \frac{x_\alpha^n}{n!}(v) \in p^{n\mathrm{ht}(\alpha)}\, L_\lambda(\mathbb{Z}, \mu + n\alpha). \tag{2}$$

If $\mathrm{ht}_\lambda(\mu+n\alpha) > r$ then $p^{n\mathrm{ht}(\alpha)}\, L_\lambda(\mathbb{Z}, \mu+n\alpha)$ clearly is contained in $L_\lambda(\mathbb{Z}, r)$. Thus, we may assume that $\mathrm{ht}_\lambda(\mu+n\alpha) \le r$. Since $\mathrm{ht}_\lambda(\mu) > r$ we obtain using equation (1)

$$r - \mathrm{ht}_\lambda(\mu+n\alpha) = r - \mathrm{ht}_\lambda(\mu) + n\mathrm{ht}(\alpha) \le n\mathrm{ht}(\alpha).$$

Thus, the definition of $L_\lambda(\mathbb{Z}, r)$ shows that $p^{n\mathrm{ht}(\alpha)}\, L_\lambda(\mathbb{Z}, \mu+n\alpha)$ is contained in $L_\lambda(\mathbb{Z}, r)$. Hence, in the second case we also obtain that $p^{n\mathrm{ht}(\alpha)} \frac{x_\alpha^n}{n!} L_\lambda(\mathbb{Z}, \mu) \subseteq L_\lambda(\mathbb{Z}, r)$.
Thus, cases 1 and 2 show that $p^{n\mathrm{ht}(\alpha)} \frac{x_\alpha^n}{n!} L_\lambda(\mathbb{Z}, r) \subseteq L_\lambda(\mathbb{Z}, r)$ and the proof of the Lemma is complete.

(1.5) Truncation of an irreducible representation. Since $L_\lambda(\mathbb{Z}, r)$ is invariant under $\mathcal{S}$ we obtain a representation

$$\rho_\lambda : \mathcal{S} \to \mathrm{End}(L_\lambda(\mathbb{Z})/L_\lambda(\mathbb{Z}, r)),$$

which we call the truncation of length r of the representation $(\rho_\lambda, L_\lambda)$. We also call the $\mathcal{S}$-module $L_\lambda(\mathbb{Z})/L_\lambda(\mathbb{Z}, r)$ the truncation of length r of the highest weight module $L_\lambda(\mathbb{Z})$. We note that

$$L_\lambda(\mathbb{Z})/L_\lambda(\mathbb{Z}, r) = \bigoplus_{\substack{\mu\in\Pi_\lambda \\ \mathrm{ht}_\lambda(\mu)\le r}} \frac{\overline{L_\lambda(\mathbb{Z}, \mu)}}{p^{r-\mathrm{ht}_\lambda(\mu)}\overline{L_\lambda(\mathbb{Z}, \mu)}}, \tag{3}$$

where the bar " $^-$ " denotes "image in $L_\lambda(\mathbb{Z})/L_\lambda(\mathbb{Z}, r)$". In particular, $L_\lambda(\mathbb{Z})/L_\lambda(\mathbb{Z}, r)$ is a finitely generated torsion $\mathbb{Z}$-module, which is annihilated by p^r.

2 Truncation of Verma modules

(2.1) Integral Verma modules and their Truncation. In this section, we introduce an analogue of the truncation of irreducible representations for Verma modules. This will be used in section 3. We let $\lambda \in \mathfrak{h}^*$ and we denote by

$$I_\lambda = \langle \mathcal{U}^+, h_\alpha - \lambda(h_\alpha),\ \alpha \in \Delta \rangle_{\mathcal{U}}$$

the left ideal (i.e. the $\mathcal{U}$-submodule) in $\mathcal{U}$ generated by $\mathcal{U}^+$ and the $h_\alpha - \lambda(h_\alpha)$ for $\alpha \in \Delta$. The Verma module of highest weight λ is defined as $V_\lambda := \mathcal{U}/I_\lambda$. Thus, V_λ is a cyclic $\mathcal{U}$-module generated by the maximal vector $\mathrm{v}_\lambda = 1 + I_\lambda$. Using the PBW Theorem we even find that $V_\lambda = \mathcal{U}^-\mathrm{v}_\lambda$ is a free $\mathcal{U}^-$-module, hence,

$$V_\lambda = \bigoplus_{\mathbf{n}\in\mathbb{N}_0^s} \mathbb{C} X_-^{\mathbf{n}} \mathrm{v}_\lambda. \tag{1}$$

We denote by $V_\lambda(\mu)$ the weight μ subspace in V_λ. Hence,

$$V_\lambda = \bigoplus_{\mu\le\lambda} V_\lambda(\mu) \tag{2}$$

and

$$V_\lambda(\mu) = \bigoplus_{\substack{\mathbf{n}\in\mathbb{N}_0^s \\ n_1\alpha_1+\cdots+n_s\alpha_s=\lambda-\mu}} \mathbb{C}X_-^{\mathbf{n}}\mathrm{v}_\lambda \tag{3}$$

(to see this, note that by equation (1) V_λ is the direct sum of the subspaces appearing on the right hand side of equation (3) and that these spaces have weight μ). From now on, we assume in addition that $\lambda \in \mathfrak{h}^*$ is integral. We define the integral Verma module as

$$V_\lambda(\mathbb{Z}) = \mathcal{U}_{\mathbb{Z}}\mathrm{v}_\lambda.$$

$V_\lambda(\mathbb{Z})$ is an admissible, i.e. $\mathcal{U}_{\mathbb{Z}}$-invariant lattice in V_λ. For any integral and dominant weight λ and any $\mathbf{n} = (n_\beta)_{\beta\in\Delta}$ we set

$$\lambda(H^{\mathbf{n}}) = \prod_{\beta\in\Delta} \binom{\lambda(h_\beta)}{n_\beta}.$$

We note that the integrality of λ implies that $\lambda(H^{\mathbf{n}}) \in \mathbb{Z}$. Since $H^{\mathbf{n}}\mathrm{v}_\lambda = \lambda(H^{\mathbf{n}})\mathrm{v}_\lambda$ and since $\mathcal{U}_{\mathbb{Z}}^+$ annihilates v_λ we obtain

$$V_\lambda(\mathbb{Z}) = \mathcal{U}_{\mathbb{Z}}^-\mathrm{v}_\lambda = \bigoplus_{\mathbf{n}\in\mathbb{N}_0^s} \mathbb{Z}X_-^{\mathbf{n}}\mathrm{v}_\lambda. \tag{4}$$

i.e. $\mathcal{V}_\lambda(\mathbb{Z})$ is a free $\mathcal{U}_{\mathbb{Z}}^-$-module. This implies in particular that

$$V_\lambda(\mathbb{Z}) = \bigoplus_{\mu\leq\lambda} V_\lambda(\mathbb{Z},\mu),$$

where

$$V_\lambda(\mathbb{Z},\mu) = \bigoplus_{\substack{\mathbf{n}\in\mathbb{N}_0^s \\ n_1\alpha_1+\cdots+n_s\alpha_s=\lambda-\mu}} \mathbb{Z}X_-^{\mathbf{n}}\mathrm{v}_\lambda = V_\lambda(\mu)\cap V_\lambda(\mathbb{Z}) \tag{5}$$

is the weight μ subspace of $V_\lambda(\mathbb{Z})$ (the last "=" in equation (5) follows from equations (3) and (4)). The set of non-trivial weights in V_λ is given as $\{\mu \in \mathfrak{h}^*,\, \mu \leq \lambda\}$ and the relative height function extends to a mapping

$$\mathrm{ht}_\lambda : \{\mu \in \mathfrak{h}^*,\, \mu \leq \lambda\} \to \mathbb{N}_0$$

($\mathrm{ht}_\lambda(\mu) = \mathrm{ht}(\lambda-\mu)$). For any non-negative integer r we define the following $\mathbb{Z}$-submodule of V_λ:

$$V_\lambda(\mathbb{Z},r) = \bigoplus_{\substack{\mu\leq\lambda \\ 0\leq \mathrm{ht}_\lambda(\mu)\leq r}} p^{r-\mathrm{ht}_\lambda(\mu)}V_\lambda(\mathbb{Z},\mu) \oplus \bigoplus_{\substack{\mu\leq\lambda \\ \mathrm{ht}(\mu)>r}} V_\lambda(\mathbb{Z},\mu).$$

Clearly, $V_\lambda(\mathbb{Z},r) \leq V_\lambda(\mathbb{Z})$ is a $\mathcal{U}_{\mathbb{Z}}^-$-submodule.

Lemma. *$V_\lambda(\mathbb{Z},r)$ is an $\mathcal{S}$-invariant subspace of $V_\lambda(\mathbb{Z})$.*

Proof. This is essentially the same proof as the one of (1.4) Lemma.

In particular, $V_\lambda(\mathbb{Z})/V_\lambda(\mathbb{Z},r)$ is an $\mathcal{S}$-module, which we call the truncation of $V_\lambda(\mathbb{Z})$ of length r.

(2.2) Relation to Truncation of Irreducible Representations. We relate the truncation of Verma modules to the truncation of finite dimensional irreducible representations of $\mathfrak{g}$. This will be used in section 3. We select an integral and dominant weight λ and for all simple roots α we set

$$m_\alpha = \lambda(h_\alpha).$$

We denote by

$$U_\lambda = \langle x_{-\alpha}^{m_\alpha+1}\mathrm{v}_\lambda,\, \alpha \in \Delta\rangle_{\mathcal{U}} = \sum_{\alpha\in\Delta} \mathcal{U}x_{-\alpha}^{m_\alpha+1}\mathrm{v}_\lambda$$

the $\mathcal{U}$-submodule of V_λ, which is generated by the elements $x_{-\alpha}^{m_\alpha+1}\mathrm{v}_\lambda$, $\alpha \in \Delta$. We claim that there is an isomorphism of $\mathcal{U}$-modules

$$\varphi :\, V_\lambda/U_\lambda \cong L_\lambda,$$

which sends $\mathrm{v}_\lambda + U_\lambda$ to v_λ. In fact using Theorem 21.4 in [Hu], p. 115 there is an isomorphism

$$\mathcal{U}/\langle I_\lambda, x_{-\alpha}^{m_\alpha+1},\, \alpha\in\Delta\rangle_{\mathcal{U}} \xrightarrow{\varphi} L_\lambda,$$

which sends the coset of $1 \in \mathcal{U}$ to v_λ. Using the isomorphism Theorem we obtain

$$L_\lambda \cong \mathcal{U}/\langle I_\lambda,\, x_{-\alpha}^{m_\alpha+1},\, \alpha\in\Delta\rangle_{\mathcal{U}} \cong \mathcal{U}/I_\lambda\Big/\langle I_\lambda, x_{-\alpha}^{m_\alpha+1},\, \alpha\in\Delta\rangle_{\mathcal{U}}/I_\lambda \overset{\psi}{\cong} V_\lambda/U_\lambda;$$

here, the isomorphism ψ is induced by the isomorphism $\mathcal{U}/I_\lambda \to L_\lambda$, $X + I_\lambda \mapsto X\mathrm{v}_\lambda$. We set

$$U_\lambda(\mathbb{Z}) = U_\lambda \cap V_\lambda(\mathbb{Z}),$$

hence, φ induces an isomorphism of $\mathcal{U}_\mathbb{Z}$-modules

$$\varphi :\, V_\lambda(\mathbb{Z})/U_\lambda(\mathbb{Z}) \to L_\lambda(\mathbb{Z}), \tag{6}$$

which is defined by sending $\mathrm{v}_\lambda + U_\lambda(\mathbb{Z})$ to v_λ. As a consequence, we obtain that

$$U_\lambda = \langle x_{-\alpha}^{m_\alpha+1},\, \alpha\in\Delta\rangle_{\mathcal{U}^-} = \sum_{\alpha\in\Delta} \mathcal{U}^- x_{-\alpha}^{m_\alpha+1}\mathrm{v}_\lambda. \tag{7}$$

In fact, the inclusion "$\supseteq$" is obvious. To prove the reverse inclusion, we let $\alpha \in \Delta$ and we denote by $S_\alpha \leq \mathfrak{g}$ the sl_2-Tripel attached to α. The isomorphism ψ induces an isomorphsm $S_\alpha v_\lambda \cong S_\alpha \mathrm{v}_\lambda/U_\lambda(\alpha)$, where $U_\lambda(\alpha) = U_\lambda \cap S_\alpha \mathrm{v}_\lambda$. If $x_\alpha x_{-\alpha}^{m_\alpha+1} v_\lambda \notin U_\lambda(\alpha)$ did not vanish, this would imply that $x_{-\alpha}^{m_\alpha} = 0$ a contradiction. Hence,

$$x_\alpha x_{-\alpha}^{m_\alpha+1}\mathrm{v}_\lambda = 0$$

for all simple roots $\alpha \in \Delta$. Since any $X \in \mathcal{U}$ is a linear combination of terms $X_-^{\mathbf{a}} H^{\mathbf{b}} X_+^{\mathbf{c}}$ we deduce that

$$U_\lambda = \sum_{\alpha\in\Delta} \mathcal{U} x_{-\alpha}^{m_\alpha+1} \mathrm{v}_\lambda = \sum_{\alpha\in\Delta} \mathcal{U}^- x_{-\alpha}^{m_\alpha+1} \mathrm{v}_\lambda$$

Lemma. *Let λ be an integral and dominant weight and assume that $m_\alpha := \langle \lambda, \alpha \rangle \geq r$ for all $\alpha \in \Delta$. Then*

$$U_\lambda(\mathbb{Z}) \leq V_\lambda(\mathbb{Z}, r)$$

is an $\mathcal{S}$-submodule and

$$\varphi(V_\lambda(\mathbb{Z}, r)/U_\lambda(\mathbb{Z})) = L_\lambda(\mathbb{Z}, r).$$

Proof. By equation (7) we know that

$$U_\lambda(\mathbb{Z}) \subseteq U_\lambda = \sum_{\alpha\in\Delta} \mathcal{U}^- x_{-\alpha}^{m_\alpha+1} \mathrm{v}_\lambda.$$

Since we assume that $m_\alpha \geq r$ we know that the weight μ of any vector v in $\mathcal{U}^- x_{-\alpha}^{m_\alpha+1} \mathrm{v}_\lambda$ has height $\mathrm{ht}_\lambda(\mu) > r$; thus we obtain

$$U_\lambda(\mathbb{Z}) \subseteq \bigoplus_{\substack{\mu\leq\lambda \\ \mathrm{ht}_\lambda(\mu)>r}} V_\lambda(\mu).$$

Since $U_\lambda(\mathbb{Z}) \subseteq V_\lambda(\mathbb{Z})$ we obtain

$$U_\lambda(\mathbb{Z}) \subseteq \Big(\bigoplus_{\substack{\mu\leq\lambda \\ \mathrm{ht}_\lambda(\mu)>r}} V_\lambda(\mu) \Big) \cap V_\lambda(\mathbb{Z}) = \bigoplus_{\substack{\mu\leq\lambda \\ \mathrm{ht}_\lambda(\mu)>r}} V_\lambda(\mathbb{Z}, \mu).$$

To prove the second statement we note that $\varphi : V_\lambda(\mathbb{Z})/U_\lambda(\mathbb{Z}) \to L_\lambda(\mathbb{Z})$ in particular is an isomorphism of $\mathcal{U}^o$-modules, hence, φ respects the weight spaces, i.e.

$$\varphi(V_\lambda(\mathbb{Z}, \mu) + U_\lambda(\mathbb{Z})) = L_\lambda(\mathbb{Z}, \mu)$$

for all integral weights μ. This implies that $\varphi(V_\lambda(\mathbb{Z}, r)/U_\lambda(\mathbb{Z})) = L_\lambda(\mathbb{Z}, r)$. Thus, the proof of the Lemma is complete.

3 Local constancy of the Truncated Representation

(3.1) Some auxiliary Lemmas. We collect some auxiliary Lemmas which will be needed in the proofs in section (3.2) of the local constancy of the truncated Verma module. For any $\mathbf{n} = (n_i)_i \in \mathbb{N}_0^s$ and any $\mathbf{m} = (m_\gamma)_{\gamma\in\Delta} \in \mathbb{N}_0^\Delta$ we set

$$X_-^{\cdot\mathbf{n}} = \prod_{i=1}^s x_{\alpha_i}^{n_i}, \quad X_+^{\cdot\mathbf{n}} = \prod_{i=1}^s x_{-\alpha_i}^{n_i}, \quad \text{and} \quad H^{\cdot\mathbf{m}} = \prod_{\gamma\in\Delta} h_\gamma^{m_\gamma}.$$

Moreover, we define the length of $\mathbf{n} = (n_i)_i \in \mathbb{N}_0^s$ as $\ell(\mathbf{n}) = \sum_i n_i$.

(3.1.1) Lemma. *Let $\alpha \in \Phi^+$ be any positive root and let $\mathbf{n} = (n_i)_i \in \mathbb{N}_0^s$. Then*

$$(1) \qquad x_\alpha X_-^{\cdot\mathbf{n}} = \sum_{\mathbf{a},\mathbf{b} \in \mathbb{N}_0^s} \zeta_{\mathbf{a},\mathbf{b}} X_-^{\cdot\mathbf{a}} X_+^{\cdot\mathbf{b}} + \sum_{\mathbf{a} \in \mathbb{N}_0^s, \gamma \in \Phi^+} \zeta_{\mathbf{a},\gamma} X_-^{\cdot\mathbf{a}} h_\gamma,$$

with certain $\zeta_{\mathbf{a},\mathbf{b}}, \zeta_{\mathbf{a},\gamma} \in \mathbb{C}$.

Proof. We denote by i the smallest index such that $n_i \neq 0$ and we set $\beta = \alpha_i$ ($\alpha_1, \ldots, \alpha_s$ is the ordered basis of Φ^+; cf. section (1.1)). Hence, we may write $X_-^{\cdot\mathbf{n}} = x_{-\beta} X_-^{\cdot\mathbf{n}'}$, where $\ell(\mathbf{n}') = \ell(\mathbf{n}) - 1$ and we obtain

$$(2) \qquad \begin{aligned} x_\alpha X_-^{\cdot\mathbf{n}} &= x_\alpha x_{-\beta} X_-^{\cdot\mathbf{n}'} \\ &= x_{-\beta} x_\alpha X_-^{\cdot\mathbf{n}'} + [x_\alpha, x_{-\beta}] X_-^{\cdot\mathbf{n}'}. \end{aligned}$$

We want to show that $[x_\alpha, x_{-\beta}] X_-^{\cdot\mathbf{n}'}$ is of the form as on the right hand side of equation (1), i.e.

$$(3) \qquad [x_\alpha, x_{-\beta}] X_-^{\cdot\mathbf{n}'} = \sum_{\mathbf{a},\mathbf{b} \in \mathbb{N}_0^s} \zeta'_{\mathbf{a},\mathbf{b}} X_-^{\cdot\mathbf{a}} X_+^{\cdot\mathbf{b}} + \sum_{\mathbf{a} \in \mathbb{N}_0^s, \gamma \in \Phi^+} \zeta'_{\mathbf{a},\gamma} X_-^{\cdot\mathbf{a}} h_\gamma$$

for certain $\zeta'_{\mathbf{a},\mathbf{b}}, \zeta_{\mathbf{a},\gamma} \in \mathbb{C}$. To this end we note that there are four possibilities for the difference $\alpha - \beta$:

- $\alpha - \beta$ is a positive root. In this case Theorem 25.2 in [Hu], p. 147 implies that $[x_\alpha, x_{-\beta}] = \zeta x_{\alpha-\beta}$ for some $\zeta \in \mathbb{Z}$
- $\alpha - \beta$ is no root, hence, $[x_\alpha, x_{-\beta}] = 0$
- $\alpha - \beta = 0$. In this case Theorem 25.2 in [Hu], p. 147 implies that $[x_\alpha, x_{-\beta}] = \sum_{\gamma \in \Delta} \zeta_\gamma h_\gamma$ for certain $\zeta_\gamma \in \mathbb{Z}$

or

- $\alpha - \beta$ is a negative root. Again, in this case Theorem 25.2 in [Hu], p. 147 implies that $[x_\alpha, x_{-\beta}] = \zeta x_{\alpha-\beta}$ for some $\zeta \in \mathbb{Z}$.

In the first case using an induction over the length of $\mathbf{n}'$, equation (1) (with the positive root α replaced by the positive root $\alpha - \beta$ and $\mathbf{n}$ replaced by $\mathbf{n}'$) shows that $[x_\alpha, x_{-\beta}] X_-^{\cdot\mathbf{n}'} = \zeta x_{\alpha-\beta} X_-^{\cdot\mathbf{n}'}$ is of the form (3). In the second case $[x_\alpha, x_{-\beta}] X_-^{\cdot\mathbf{n}'}$ vanishes, hence equation (3) holds trivially. In the third case for any positive root γ we have $h_\alpha x_{-\gamma} = x_{-\gamma} h_\alpha + [h_\alpha, x_{-\gamma}] = x_{-\gamma} h_\alpha - \gamma(h_\alpha) x_{-\gamma}$, where $\gamma(h_\alpha) \in \mathbb{Z}$. A simple induction over the length of $\mathbf{n}'$ therefore shows that

$$h_\alpha X_-^{\cdot\mathbf{n}'} = X_-^{\cdot\mathbf{n}'} h_\alpha + \zeta X_-^{\cdot\mathbf{n}'}$$

for some $\zeta \in \mathbb{C}$, hence, equation (3) holds. In the fourth case Lemma 5.14 in [Ha], p. 135 implies that $[x_\alpha, x_{-\beta}] X_-^{\cdot\mathbf{n}'} = \zeta x_{\alpha-\beta} X_-^{\cdot\mathbf{n}'}$ can be written

$$[x_\alpha, x_{-\beta}] X_-^{\cdot\mathbf{n}'} = \sum_{\mathbf{a} \in \mathbb{N}_0^s} \zeta_{\mathbf{a}} X_-^{\cdot\mathbf{a}}$$

for some $\zeta_{\mathbf{a}} \in \mathbb{C}$. Thus, equation (3) is proven and we have shown that

$$x_\alpha X_-^{\cdot\mathbf{n}} = x_{-\beta} x_\alpha X_-^{\cdot\mathbf{n}'} + \text{terms as in the right hand side of eq. (1)},$$

where now $\ell(\mathbf{n}') < \ell(\mathbf{n})$. Repeating the above computation we find

$$x_\alpha X_-^{\cdot\mathbf{n}'} = x_{-\beta'} x_\alpha X_-^{\cdot\mathbf{n}''} + \text{terms as in the right hand side of eq. (1)},$$

where now $\ell(\mathbf{n}'') < \ell(\mathbf{n}')$ and $\beta' = \alpha_i$, i the smallest index such that $n_i' \neq 0$. Proceeding in this way we finally obtain that

$$x_\alpha X_-^{\cdot\mathbf{n}} = X_-^{\cdot\mathbf{n}} x_{-\alpha} + \text{terms as in the right hand side of eq. (1)}.$$

This, is the claim and the proof of the Lemma is complete.

We define the length of $\mathbf{m} = (m_\gamma)_\gamma \in \mathbb{N}_0^\Delta$ as $\ell(\mathbf{m}) = \sum_{\gamma\in\Delta} m_\gamma$.

(3.1.2) Lemma. *For any positive root α and any $k \in \mathbb{N}$*

$$x_\alpha^k X_-^{\cdot\mathbf{n}} = \sum_{\substack{\mathbf{a},\mathbf{c}\in\mathbb{N}_0^s,\, \mathbf{b}\in\mathbb{N}_0^\Delta \\ \ell(\mathbf{b})\le k}} \zeta_{\mathbf{a},\mathbf{b},\mathbf{c}} X_-^{\cdot\mathbf{a}} H^{\cdot\mathbf{b}} X_+^{\cdot\mathbf{c}}.$$

Proof. We use induction on k. The case $k = 1$ is immediate by the preceeding Lemma. We write using the induction hypothesis

$$x_\alpha^{k+1} X_-^{\cdot\mathbf{n}} = x_\alpha x_\alpha^k X_-^{\cdot\mathbf{n}} = x_\alpha \Big(\sum_{\substack{\mathbf{a},\mathbf{c}\in\mathbb{N}_0^s,\, \mathbf{b}\in\mathbb{N}_0^\Delta \\ \ell(\mathbf{b})\le k}} \zeta_{\mathbf{a},\mathbf{b},\mathbf{c}} X_-^{\cdot\mathbf{a}} H^{\cdot\mathbf{b}} X_+^{\cdot\mathbf{c}} \Big).$$

Using the expression for $x_\alpha X_-^{\cdot\mathbf{a}}$ from the preceding Lemma we obtain

$$x_\alpha^{k+1} X_-^{\cdot\mathbf{n}} = \sum_{\substack{\mathbf{a},\mathbf{c}\in\mathbb{N}_0^s,\, \mathbf{b}\in\mathbb{N}_0^\Delta \\ \ell(\mathbf{b})\le k}} \zeta_{\mathbf{a},\mathbf{b},\mathbf{c}} \sum_{\mathbf{a}',\mathbf{b}'} \zeta_{\mathbf{a}',\mathbf{b}'} X_-^{\cdot\mathbf{a}'} X_+^{\cdot\mathbf{b}'} H^{\cdot\mathbf{b}} X_+^{\cdot\mathbf{c}} + \sum_{\mathbf{a}',\gamma\in\Phi^+} X_-^{\cdot\mathbf{a}'} h_\gamma H^{\cdot\mathbf{b}} X_+^{\cdot\mathbf{c}}. \tag{4}$$

Taking into account that for any positive root γ

$$X_+^{\cdot\mathbf{b}'} h_\gamma = h_\gamma X_+^{\cdot\mathbf{b}'} + \zeta X_+^{\cdot\mathbf{b}'}$$

for some $\zeta \in \mathbb{C}$ we see that $X_+^{\cdot\mathbf{b}'} H^{\cdot\mathbf{b}}$ can be written as a sum $\sum_{\mathbf{n},\, \ell(\mathbf{n})\le\ell(\mathbf{b})} \zeta_\mathbf{n} H^{\cdot\mathbf{n}} X_+^{\cdot\mathbf{b}'}$; hence, the summation over $\mathbf{a}', \mathbf{b}'$ in equation (4) is of the form as claimed in the Lemma. Since $\mathcal{U}^o$ is abelian we see that the summation over $\mathbf{a}', \gamma$ in equation (4) also is of the form as claimed in the lemma. This completes the proof of the Lemma.

(3.1.3) Lemma. *For any positive root $\alpha \in \Phi^+$ and any $k \in \mathbb{N}$*

$$\frac{x_\alpha^k}{k!} X_-^{\mathbf{n}} = \sum_{\substack{\mathbf{a},\mathbf{b},\mathbf{c}\in\mathbb{N}_0^s \\ \ell(\mathbf{b})\le k}} \zeta_{\mathbf{a},\mathbf{b},\mathbf{c}} X_-^{\mathbf{a}} H^{\mathbf{b}} X_+^{\mathbf{c}}.$$

with certain $\zeta_{\mathbf{a},\mathbf{b},\mathbf{c}} \in \mathbb{Z}$.

Proof. The $\mathbb{C}$-linear span $\langle H^{\cdot\mathbf{n}},\, \ell(\mathbf{n}) \le k\rangle_\mathbb{C}$ contains the $\mathbb{C}$-linear span $\langle H^{\mathbf{n}},\, \ell(\mathbf{n}) \le k\rangle_\mathbb{C}$ and since $H^\mathbf{n}$ has "leading monomial" $\prod_{\gamma\in\Delta} h_\gamma^{n_\gamma} = H^{\cdot\mathbf{n}}$ we find that

$$\langle H^{\cdot\mathbf{n}},\, \ell(\mathbf{n}) \le k\rangle_\mathbb{C} = \langle H^{\mathbf{n}},\, \ell(\mathbf{n}) \le k\rangle_\mathbb{C}.$$

Thus, using the preceding Lemma we deduce that we can write

$$\frac{x_\alpha^k}{k!}X_-^{\mathbf{n}} = \sum_{\substack{\mathbf{a},\mathbf{b},\mathbf{c}\in\mathbb{N}_0^s\\ \ell(\mathbf{b})\le k}} \zeta_{\mathbf{a},\mathbf{b},\mathbf{c}} X_-^{\mathbf{a}} H^{\mathbf{b}} X_+^{\mathbf{c}}$$

with certain $\zeta_{\mathbf{a},\mathbf{b},\mathbf{c}} \in \mathbb{C}$. On the other hand, $\frac{x_\alpha^k}{k!}X_-^{\mathbf{n}}$ is contained in the $\mathbb{Z}$-lattice $\mathcal{U}_\mathbb{Z}$ of $\mathcal{U}$ and since $\mathcal{U}_\mathbb{Z}$ is a free $\mathbb{Z}$-module with basis $X_-^{\mathbf{a}} H^{\mathbf{b}} X_+^{\mathbf{c}}$ we deduce that $\zeta_{\mathbf{a},\mathbf{b},\mathbf{c}} \in \mathbb{Z}$. Thus, the Lemma is proven.

(3.2) Local Constancy of truncated Verma modules. The following Proposition shows that the truncations of two Verma modules $V_\lambda(\mathbb{Z})$ and $V_{\lambda'}(\mathbb{Z})$ are isomorphic if the highest weights λ and λ' are sufficiently close in the p-adic sense.

Proposition. *Let $\lambda, \lambda' \in \Gamma_{\rm sc}$ be integral weights satisfying $\lambda \equiv \lambda' \pmod{p^{2r}\Gamma_{\rm ad}}$. Then*

$$V_\lambda(\mathbb{Z})/V_\lambda(\mathbb{Z}, r) \cong V_{\lambda'}(\mathbb{Z})/V_{\lambda'}(\mathbb{Z}, r)$$

as $\mathcal{S}$-modules.

Proof. We write $V_\lambda(\mathbb{Z}) = \mathcal{U}_\mathbb{Z}^- \mathrm{v}_\lambda$ and $V_{\lambda'}(\mathbb{Z}) = \mathcal{U}_\mathbb{Z}^- \mathrm{v}_{\lambda'}$. Since $V_\lambda(\mathbb{Z})$ is a free $\mathcal{U}_\mathbb{Z}^-$-module, there is a uniquely determined isomorphism of $\mathcal{U}_\mathbb{Z}^-$-modules

$$\Phi:\ V_\lambda(\mathbb{Z}) \to V_{\lambda'}(\mathbb{Z}), \tag{5}$$

which sends v_λ to $\mathrm{v}_{\lambda'}$, hence, $\Phi(X\mathrm{v}_\lambda) = X\mathrm{v}_{\lambda'}$ for all $X \in \mathcal{U}_\mathbb{Z}^-$. We claim that

$$\Phi(V_\lambda(\mathbb{Z}, r)) = V_{\lambda'}(\mathbb{Z}, r). \tag{6}$$

It is sufficient to prove the inclusion "$\subseteq$", the proof of the reverse inclusion is analogous. We select an arbitrary weight $\mu \le \lambda$ and a vector $v \in V_\lambda(\mathbb{Z}, \mu)$. In view of equation (5) in section (2.1), v can be written $v = \sum_{\mathbf{n}} c_{\mathbf{n}} X_-^{\mathbf{n}} \mathrm{v}_\lambda$, where $c_{\mathbf{n}} \in \mathbb{Z}$ and the sum runs over all $\mathbf{n} = (n_i) \in \mathbb{N}_0^s$ satisfying $\sum_{i=1}^s n_i \alpha_i = \lambda - \mu$. Applying Φ yields $\Phi(v) = \sum_{\mathbf{n}} c_{\mathbf{n}} X_-^{\mathbf{n}} \mathrm{v}_{\lambda'}$, hence, $\Phi(v)$ is contained in $V_{\lambda'}(\mathbb{Z}, \lambda' - (\lambda - \mu))$ and we have shown that $\Phi(V_\lambda(\mathbb{Z}, \mu)) \subseteq V_{\lambda'}(\mathbb{Z}, \lambda' - (\lambda - \mu))$. Since

$$\mathrm{ht}_{\lambda'}(\lambda' - (\lambda - \mu)) = \lambda - \mu = \mathrm{ht}_\lambda(\lambda - (\lambda - \mu)) = \mathrm{ht}_\lambda(\mu)$$

and μ was arbitrary, we thus have shown that

$$\Phi\Big(\sum_{\mu\le\lambda,\,\mathrm{ht}_\lambda(\mu)=s} V_\lambda(\mathbb{Z}, \mu)\Big) \subseteq \sum_{\mu\le\lambda,\,\mathrm{ht}_{\lambda'}(\mu)=s} V_{\lambda'}(\mathbb{Z}, \mu). \tag{7}$$

By the definition of $V_\lambda(\mathbb{Z}, r)$ this implies that "$\subseteq$" in equation (6) and, hence, equation (6) itself holds. In particular, Φ induces an isomorphism of $\mathcal{U}_\mathbb{Z}^-$-modules

$$\Phi:\ V_\lambda(\mathbb{Z})/V_\lambda(\mathbb{Z}, r) \to V_{\lambda'}(\mathbb{Z})/V_{\lambda'}(\mathbb{Z}, r),$$

i.e. Φ commutes with all $\frac{x_{-\alpha}^n}{n!}$, $\alpha \in \Phi^+$. It remains to show that Φ commutes with the action of all generators "$p^{n\mathrm{ht}(\alpha)}\frac{x_\alpha^n}{n!}$" of $\mathcal{S}$, i.e.

$$\Phi(t^n \frac{x_\alpha^n}{n!} v) = t^n \frac{x_\alpha^n}{n!} \Phi(v) \tag{8}$$

for all $\alpha \in \Phi^+$, $t \in p^{\mathrm{ht}(\alpha)}\mathbb{Z}$, $v \in V_\lambda(\mathbb{Z})/V_\lambda(\mathbb{Z}, r)$ and $n \geq 0$ (this implies that Φ is an isomorphism of $\mathcal{S}$-modules as claimed; note that the truncations are $\mathcal{S}$-modules by (2.1) Lemma). Since Φ is $\mathbb{Z}$-linear we may assume that v is of the form

$$v = X_-^{\mathbf{n}} \mathrm{v}_\lambda + L_\lambda(\mathbb{Z}, r) = X_-^{\mathbf{n}} \bar{\mathrm{v}}_\lambda \qquad (\bar{\mathrm{v}}_\lambda = \mathrm{v}_\lambda + V_\lambda(\mathbb{Z}, r))$$

for some $\mathbf{n} \in \mathbb{N}_0^s$. Since p divides $p^{\mathrm{ht}(\alpha)}$, it divides t and since p^r annihilates $V_\lambda(\mathbb{Z})/V_\lambda(\mathbb{Z}, r)$ and $V_{\lambda'}(\mathbb{Z})/V_{\lambda'}(\mathbb{Z}, r)$, we see that both sides of equation (8) vanish if $n \geq r$. We therefore may assume that

$$n < r.$$

Using (3.1.3) Lemma we can write

$$\frac{x_\alpha^n}{n!} X_-^{\mathbf{n}} = \sum_{\substack{\mathbf{a},\mathbf{b},\mathbf{c} \\ \ell(\mathbf{b}) \leq n}} \zeta_{\mathbf{a},\mathbf{b},\mathbf{c}} X_-^{\mathbf{a}} H^{\mathbf{b}} X_+^{\mathbf{c}}, \tag{9}$$

where $\zeta_{\mathbf{a},\mathbf{b},\mathbf{c}} \in \mathbb{Z}$ and $\ell(\mathbf{b}) \leq n < r$. We set

$$\lambda(H^{\mathbf{b}}) = \prod_{\alpha \in \Delta} \binom{\lambda(h_\alpha)}{b_\alpha} \in \mathbb{Z}.$$

We note that the integrality of λ implies that $\lambda(H^{\mathbf{b}}) \in \mathbb{Z}$. Since $X_+^{\mathbf{n}}$ annihilates v_λ and since

$$\binom{h_\alpha}{b_\alpha} \mathrm{v}_\lambda = \binom{\lambda(h_\alpha)}{b_\alpha} \mathrm{v}_\lambda$$

for all simple roots α, we obtain using equation (9)

$$\begin{aligned} \Phi(t^n \frac{x_\alpha^n}{n!} X_-^{\mathbf{n}} \bar{\mathrm{v}}_\lambda) &= \Phi(t^n \sum_{\mathbf{a},\mathbf{b}} \zeta_{\mathbf{a},\mathbf{b},0}\, \lambda(H^{\mathbf{b}}) X_-^{\mathbf{a}} \bar{\mathrm{v}}_\lambda \\ &= t^n \sum_{\mathbf{a},\mathbf{b}} \zeta_{\mathbf{a},\mathbf{b},0}\, \lambda(H^{\mathbf{b}}) X_-^{\mathbf{a}} \bar{\mathrm{v}}_{\lambda'}. \end{aligned} \tag{10}$$

In the same way we find

$$t^n \frac{x_\alpha^n}{n!} \Phi(X_-^{\mathbf{n}} \bar{\mathrm{v}}_\lambda) = t^n \frac{x_\alpha^n}{n!} X_-^{\mathbf{n}} \bar{\mathrm{v}}_{\lambda'} = t^n \sum_{\mathbf{a},\mathbf{b}} \zeta_{\mathbf{a},\mathbf{b},0} \lambda'(H^{\mathbf{b}}) X_-^{\mathbf{a}} \bar{\mathrm{v}}_{\lambda'}. \tag{11}$$

For any root α we know that $\alpha(h_\beta) = \langle \alpha, \beta \rangle \in \mathbb{Z}$ by the defining properties of root systems (cf. [Hu], equation (R4), p. 42), hence, $\mu(h_\beta) \in \mathbb{Z}$ for all $\mu \in \Gamma_{\mathrm{ad}}$ and all $\beta \in \Delta$. The congruence $\lambda \equiv \lambda' \pmod{p^{2r}\Gamma_{\mathrm{ad}}}$ therefore implies that

$$\lambda(h_\beta) - \lambda'(h_\beta) \in p^{2r}\mathbb{Z}. \tag{12}$$

On the other hand, since $\ell(\mathbf{b}) < r$ we know that $b_\beta < r$ for all $\beta \in \Delta$. Hence, $v_p(b_\beta!) < r$ and together with equation (12) we find

$$v_p\left(\binom{\lambda(h_\beta)}{b_\beta} - \binom{\lambda'(h_\beta)}{b_\beta}\right) \geq r$$

for all $\beta \in \Delta$, hence,

$$\lambda(H^{\mathbf{b}}) \equiv \lambda'(H^{\mathbf{b}}) \pmod{p^r}.$$

Together with equations (10) and (11) and taking into account that p^r annihilates $V_{\lambda'}(\mathbb{Z})/V_{\lambda'}(\mathbb{Z}, r)$ this finally yields

$$\Phi(t^n \frac{x_\alpha^n}{n!} X_-^{\mathbf{n}} \bar{v}_\lambda) = t^n \frac{x_\alpha^n}{n!} \Phi(X_-^{\mathbf{n}} \bar{v}_\lambda).$$

Thus, Φ is a $\mathcal{S}$-invariant and the proof of the Proposition is complete.

(3.3) Constancy of truncated finite dimensional irreducible representations.

Proposition. *Let $\lambda, \lambda' \in \Gamma_{\mathrm{sc}}$ be integral and dominant weight satisfying*

- $m_\alpha = \langle \lambda, \alpha \rangle \geq r$ *for all* $\alpha \in \Delta$ *and* $m'_\alpha = \langle \lambda', \alpha \rangle \geq$ *for all* $\alpha \in \Delta$
- $\lambda \equiv \lambda' \pmod{p^{2r}\Gamma_{\mathrm{ad}}}$.

Then

$$L_\lambda(\mathbb{Z})/L_\lambda(\mathbb{Z}, r) \cong L_{\lambda'}(\mathbb{Z})/L_{\lambda'}(\mathbb{Z}, r)$$

as $\mathcal{S}$-modules.

Proof. (2.2) Lemma implies that $U_\lambda(\mathbb{Z}) \leq V_\lambda(\mathbb{Z}, r)$; the Isomorphism Theorem then yields an isomorphism of $\mathcal{S}$-modules

$$V_\lambda(\mathbb{Z})/V_\lambda(\mathbb{Z}, r) \cong V_\lambda(\mathbb{Z})/U_\lambda(\mathbb{Z}) \Big/ V_\lambda(\mathbb{Z}, r)/U_\lambda(\mathbb{Z})$$

Again (2.2) Lemma then implies that φ (cf. equation (6) in section (2.2)) induces an isomorphism of $\mathcal{S}$-modules

$$V_\lambda(\mathbb{Z})/U_\lambda(\mathbb{Z}) \Big/ V_\lambda(\mathbb{Z}, r)/U_\lambda(\mathbb{Z}) \cong L_\lambda(\mathbb{Z})/L_\lambda(\mathbb{Z}, r).$$

Thus,

$$L_\lambda(\mathbb{Z})/L_\lambda(\mathbb{Z}, r) \cong V_\lambda(\mathbb{Z})/V_\lambda(\mathbb{Z}, r)$$

and the claim follows from (3.2) Proposition. This completes the proof of the Proposition.

4 Representations of split semi simple groups

(4.1) Chevalley groups. We fix a dominant and integral weight $\lambda_0 \in \Gamma_{\rm sc}$ such that the representation $(\rho_{\lambda_0}, L_{\lambda_0})$ of $\mathfrak{g}$ is faithful (we can always assume this by omitting the simple factors in the kernel of ρ_{λ_0}). Let R denote any $\mathbb{Z}$-algebra. For any root $\alpha \in \Phi$ and any $t \in R$ we set

$$x_\alpha(t) = \exp \rho_{\lambda_0}(tx_\alpha) \in \mathrm{SL}(L_{\lambda_0}(R)).$$

We then denote by

$$G_{\lambda_0,R} = \langle x_\alpha(t_\alpha),\ \alpha \in \Phi,\ t_\alpha \in R\rangle \ \leq \mathrm{SL}(L_{\lambda_0}(R)) \tag{1}$$

the *Chevalley group* attached to ρ_{λ_0} and R, i.e. $G_{\lambda_0,R}$ is the subgroup of $\mathrm{SL}(L_{\lambda_0}(R))$, which is generated by the $x_\alpha(t)$, $\alpha \in \Phi$, $t \in R$. Equation (1) even defines an algebraic group in the following sense: we choose a $\mathbb{Z}$-basis of $L_{\lambda_0}(\mathbb{Z})$, which yields an identification

$$\mathrm{SL}(L_{\lambda_0}(R)) = \mathrm{SL}_m(R) \qquad (m = \dim L_{\lambda_0}).$$

Hence, $G_{\lambda_0,R} \leq \mathrm{SL}_m(R)$ and there is an algebraic group $\mathbf{G}_{\lambda_0} \leq \mathbf{SL}_{m/\mathbb{Q}}$, which is defined over $\mathbb{Q}$, such that

$$\mathbf{G}_{\lambda_0}(\bar{\mathbb{Q}}) = G_{\lambda_0,\bar{\mathbb{Q}}}.$$

The group $\mathbf{G}_{\lambda_0}$ has a natural $\mathbb{Z}$-structure $\mathbf{G}_{\lambda_0/\mathbb{Z}}$ (i.e. a $\mathbb{Z}$-form of the coordinate algebra $\mathbb{Q}[\mathbf{G}_{\lambda_0}]$), such that $\mathbf{G}_{\lambda_0/\mathbb{Z}}$ embeds as a closed subscheme of $\mathbf{SL}_{m/\mathbb{Z}}$ (cf. [B], sec. 3.4, p. 18). Hence, if R is any $\mathbb{Z}$-algebra, which is embedded in $\bar{\mathbb{Q}}$, then

$$\mathbf{G}_{\lambda_0}(R) = \mathbf{G}_{\lambda_0}(\bar{\mathbb{Q}}) \cap \mathrm{SL}_m(R).$$

In particular, we obtain

$$G_{\lambda_0,R} \subseteq \mathrm{SL}(L_{\lambda_0}(R)) \cap G_{\lambda_0,\bar{\mathbb{Q}}} = \mathrm{SL}_m(R) \cap \mathbf{G}_{\lambda_0}(\bar{\mathbb{Q}}) = \mathbf{G}_{\lambda_0}(R). \tag{2}$$

The group $\mathbf{G}_{\lambda_0}$ even is split over $\mathbb{Q}$ and it is well known that any semi-simple, $\mathbb{Q}$-split algebraic group $\mathbf{G}$ is isomorphic to $\mathbf{G}_{\lambda_0}$ for some faithful representation ρ_{λ_0} of the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$.

(4.2) For any prime $p \in \mathbb{N}$ we define the level subgroup

$$K_*(p) \leq \mathbf{G}_{\lambda_0,\mathbb{Z}_p}$$

as the subgroup, which is generated by the elements $x_\alpha(t_\alpha)$, where $\alpha \in \Phi$ and

$$t_\alpha \in \begin{cases} p^{\mathrm{ht}(\alpha)}\mathbb{Z}_p & \text{if} \quad \alpha \in \Phi^+ \\ \mathbb{Z}_p & \text{if} \quad \alpha \in \Phi^-. \end{cases}$$

Equation (2) implies that

$$K_*(p) \leq \mathbf{G}_{\lambda_0}(\mathbb{Z}_p).$$

Example. We set $\mathfrak{g} = \mathfrak{sl}_n$ and we denote by $\mathfrak{h}$ the Cartan subalgebra consisting of diagonal matrices in $\mathfrak{g}$. The roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ are α_{ij}, $i \neq j$, where $\alpha_{ij}(\mathrm{diag}(h_1, \ldots, h_n)) =$

$h_i - h_j$. The set $\Delta = \{\alpha_{i+1,i},\ i = 1, \ldots, n-1\}$ is a basis for Φ and the root space $\mathfrak{g}(\alpha_{ij})$ is generated by the elementary matrix $x_{\alpha_{ij}} = e_{ij} = (\delta_{ij})_{ij}$. The height of a root is explicitly given as $\mathrm{ht}(\alpha_{ij}) = i - j$. We fix the standard matrix representation $\rho_0 : \mathfrak{g} \to \mathfrak{sl}_n$ of $\mathfrak{g}$, i.e. $\rho_0(x) = x$ for all $x \in \mathfrak{g}$. An easy computation shows that

$$e_{ij}(t) := x_{\alpha_{ij}}(t) = \exp \rho_0(te_{ij}) = 1 + te_{ij} \tag{3}$$

for all $i \neq j$. This implies in particular that $\mathbf{G}_{\rho_0} = \mathbf{SL}_n$. We *claim* that

$$K_*(p) = \{(\gamma_{ij}) \in \mathrm{SL}_n(\mathbb{Z}_p) : \gamma_{ii} \in 1 + p\mathbb{Z}_p \text{ for all } i,\ \ \gamma_{ij} \in p^{i-j}\mathbb{Z}_p \text{ for all } i > j\}. \tag{4}$$

Proof. It is easily verified that the right hand side defines a subgroup of $\mathrm{SL}_n(\mathbb{Z}_p)$ and equation (3) implies that all generators $x_{\alpha_{ij}}(t) = e_{ij}(t)$ of $K_*(p)$ (i.e. $t \in \mathbb{Z}_p$ if $i < j$ and $t \in p^{\mathrm{ht}(\alpha_{ij})}\mathbb{Z}_p = p^{i-j}\mathbb{Z}_p$ if $i > j$) are contained in the right hand side of equation (4). Hence, the inclusion "$\subseteq$" holds. To prove the reverse inclusion "$\supseteq$" let γ be contained in the right hand side of equation (4). Using left multiplication by elements $x_{\alpha_{ij}}(t)$ with $i > j$ and $t \in p^{i-j}\mathbb{Z}_p$, i.e. by using row operations, we can transform γ into an upper triangular matrix $\gamma' = (\gamma'_{ij})$ (note that $\gamma_{ij} \in p^{i-j}\mathbb{Z}_p$ if $i > j$). Since γ' still has determinant equal to 1 we know that $\prod_i \gamma'_{ii} = 1$. The SL_2-relation

$$\begin{pmatrix} t^{-1} & \\ & t \end{pmatrix} = \begin{pmatrix} 1 & \\ (t-1)t & 1 \end{pmatrix} \begin{pmatrix} 1 & t^{-1} \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ -(t-1) & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ & 1 \end{pmatrix}$$

implies that $K_*(p)$ contains the matrices

$$h_{i,i+1}(t) = \begin{pmatrix} \ddots & & & \\ & t & & \\ & & t^{-1} & \\ & & & \ddots \end{pmatrix},$$

for all $t \in 1 + p\mathbb{Z}_p$, where t appears in the i-th position. Multiplying $\gamma' = (\gamma'_{ij})$ by suitable elements $h_{i,i+1}(t)$ with $t \in 1 + p\mathbb{Z}_p$, we can transform γ' into a matrix $\gamma'' = (\gamma''_{ij})$, whose diagonal entries all equal 1, i.e. γ'' is an upper unipotent matrix. Using left multiplication by elements $x_{\alpha_{ij}}(t)$ with $i < j$ and $t \in \mathbb{Z}_p$ suitably chosen, we see that γ'' can be transformed into the unit matrix. Since the transforming elements $x_{\alpha_{ij}}(t)$ and $h_{i+1,i}(t)$, which we used, are all contained in $K_*(p)$, this finally shows that γ is contained in $K_*(p)$. Hence, the inclusion "$\supseteq$" holds and the claim is proven.

(4.3) Irreducible representations of split semi-simple groups. We fix a semi-simple algebraic $\mathbb{Q}$-group $\mathbf{G}$, which is split over $\mathbb{Q}$. Hence, $\mathbf{G} = \mathbf{G}_{\lambda_0}$ for some finite dimensional, irreducible representation $(\rho_{\lambda_0}, L_{\lambda_0})$ of the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$. We recall the description of the irreducible representations of $\mathbf{G}$. If $K/\mathbb{Q}$ is an arbitrary extension field we define for $t \in K^*$ and $\alpha \in \Delta$ the elements $h_\alpha(t)$ as in [B], 3.2 (1), p. 13. The algebraic group $\mathbf{G}$ contains a maximal torus $\mathbf{T}$ such that for every extension $K/\mathbb{Q}$ the group of K-rational points $\mathbf{T}(K)$ is generated by the elements

$$h_\alpha(t), \quad t \in K^*,\ \alpha \in \Delta$$

(cf. [B] 3.2 (1), p. 13). Moreover, $\mathbf{T}$ is defined over $\mathbb{Q}$ and splits over $\mathbb{Q}$ (cf. [B], 3.3 (3), p. 15). To any $\lambda \in \Gamma_{\lambda_0}$ $(= \langle \Pi_{\lambda_0} \rangle_{\mathbb{Z}})$ we attach a character $\lambda^\circ \in X(\mathbf{T}(\bar{\mathbb{Q}}))$ by setting

$$(5) \qquad \lambda^\circ(\prod_{\alpha\in\Delta} h_\alpha(t_\alpha)) = \prod_{\alpha\in\Delta} t_\alpha^{\lambda(h_\alpha)}$$

for all $t_\alpha \in \bar{\mathbb{Q}}^*$. In particular, λ° defines an algebraic character of $\mathbf{T}$ and the assignment $\lambda \mapsto \lambda^\circ$ defines an isomorphism

$$(6) \qquad \varphi : \Gamma_{\lambda_0} \to \mathbf{Mor}_{\mathrm{alg}}(\mathbf{T}, \mathbb{G}_m)$$

(cf. [B], 3.3 (3), p. 15). We note that μ° is in fact the "exponential" of the weight $\mu \in \Pi_{\lambda_0}$: for $h = \prod_{\alpha\in\Delta} h_\alpha(t_\alpha) \in \mathbf{T}(\bar{\mathbb{Q}})$ and any $v_\mu \in L_{\lambda_0}(\mu)$ it holds that

$$(7) \qquad hv_\mu = \mu^\circ(h)v_\mu$$

(cf. [B] 3.2 (1), p. 13).

We select an integral and dominant weight $\lambda \in \Gamma_{\mathrm{sc}}$ such that

$$\Gamma_{\lambda_0} \supseteq \Gamma_\lambda.$$

For any root $\alpha \in \Phi$ and any $t \in \bar{\mathbb{Q}}$ we set

$$(8) \qquad x_\alpha^{\rho_\lambda}(t) = \exp t\rho_\lambda(x_\alpha) \in \mathrm{SL}(L_\lambda(\bar{\mathbb{Q}})).$$

There is a surjective morphism of groups

$$\pi_{\lambda^\circ,\bar{\mathbb{Q}}} : G_{\lambda_0,\bar{\mathbb{Q}}} \to G_{\lambda,\bar{\mathbb{Q}}} \subseteq \mathrm{SL}(L_\lambda(\bar{\mathbb{Q}})),$$

which sends $x_\alpha(t)$ to $x_\alpha^{\rho_\lambda}(t)$ for all $\alpha \in \Phi$ and $t \in \bar{\mathbb{Q}}$ (cf. [B], 3.2 (4), p. 14). Hence, we obtain a representation

$$(9) \qquad \pi_{\lambda^\circ,\bar{\mathbb{Q}}} : G_{\lambda_0,\bar{\mathbb{Q}}} \to \mathrm{SL}(L_\lambda(\bar{\mathbb{Q}})).$$

We denote by $\mathbf{SL}(L_\lambda(\mathbb{Z}))$ the algebraic $\mathbb{Z}$-group, which is defined by $\mathbf{SL}(L_\lambda(\mathbb{Z}))(R) = \mathbf{SL}(L_\lambda(R))$, R any $\mathbb{Z}$-algebra. The choice of a $\mathbb{Z}$-basis of $L_\lambda(\mathbb{Z})$ yields an identification $\mathbf{SL}(L_\lambda(\mathbb{Z})) = \mathbf{SL}_m/\mathbb{Z}$, where $m = \dim L_\lambda(\mathbb{Z})$. We recall that $\mathbf{G}_{\lambda_0}$ has a natural $\mathbb{Z}$-structure. Following [B], section 4.3, p. 22, equation (9) defines a representation of the algebraic group $\mathbf{G} = \mathbf{G}_{\lambda_0}$, i.e. there is a morphism of algebraic groups

$$(10) \qquad \pi_{\lambda^\circ} : \mathbf{G}_{/\mathbb{Z}} \to \mathbf{SL}(L_\lambda(\mathbb{Z})) = \mathbf{SL}_m/\mathbb{Z},$$

which is defined over $\mathbb{Z}$ with respect to the natural $\mathbb{Z}$-structure on $\mathbf{G}_{\lambda_0}$. On $\bar{\mathbb{Q}}$-points π_{λ° is given by

$$(11) \qquad \pi_{\lambda^\circ}(x_\alpha(t)) = x_\alpha^{\rho_\lambda}(t) \in \mathrm{SL}(L_\lambda(\bar{\mathbb{Q}})) \qquad (\alpha \in \Phi, t \in \bar{\mathbb{Q}}).$$

π_{λ° is an irreducible representation of $\mathbf{G}$ with highest weight $\lambda^\circ \in X(\mathbf{T})$ and any irreducible representation of $\mathbf{G}$ is isomorphic to some π_{λ° (note that if (π_λ, L_λ) is the irreducible

representation of highest weight λ of $\mathbf{G}_{\lambda_0}$ then necessarily $\Gamma_{\lambda_0} \supseteq \Gamma_\lambda$). We note that equation (10) implies that π_{λ° defines a representation on R-points

$$\pi_{\lambda^\circ} : \mathbf{G}(R) \to \mathrm{SL}(L_\lambda(R)) = \mathrm{SL}_m(R) \tag{12}$$

for any $\mathbb{Z}$-algebra R. In particular, $G_{\lambda_0,R}$ leaves $L_\lambda(R)$ invariant, i.e. $L_\lambda(R)$ is a $G_{\lambda_0,R}$-module.

(4.4) The Truncation of an irreducible representation of a split, semi-simple group.

(4.4.1) As in section (4.3) we let $\mathbf{G}$ be an arbitrary $\mathbb{Q}$-split, semi-simple algebraic group, hence, $\mathbf{G} = \mathbf{G}_{\lambda_0}$ for some dominant and integral weight λ_0. Moreover, we fix a prime $p \in \mathbb{N}$ and we let $\lambda \in \Gamma_{\mathrm{sc}}$ be an integral and dominant weight such that

$$\Gamma_{\lambda_0} \supseteq \Gamma_\lambda. \tag{13}$$

Equation (12) then yields a representation on $\mathbb{Z}_p$-points

$$\pi_{\lambda^\circ} : \mathbf{G}(\mathbb{Z}_p) \to \mathrm{SL}(L_\lambda(\mathbb{Z}_p)).$$

We set $\mathcal{S}_p = \mathbb{Z}_p \otimes \mathcal{S}$. Clearly, $L_\lambda(\mathbb{Z}_p)$ is a $\mathcal{S}_p$-module and (1.4) Lemma implies that $L_\lambda(\mathbb{Z}_p, r)$ is a $\mathcal{S}_p$-invariant submodule. In particular, $L_\lambda(\mathbb{Z}_p)/L_\lambda(\mathbb{Z}_p, r)$ is a $\mathcal{S}_p$-module. We note that - $L_\lambda(\mathbb{Z}_p)/L_\lambda(\mathbb{Z}_p, r) \cong \mathbb{Z}_p \otimes (L_\lambda(\mathbb{Z})/L_\lambda(\mathbb{Z}, r))$ as $\mathbb{Z}_p$-modules - tensoring the isomorphism in (3.3) Proposition, which is induced by sending v_λ to $v_{\lambda'}$, with $\mathbb{Z}_p$ yields an isomorphism of $\mathbb{Z}_p$-modules $L_\lambda(\mathbb{Z})/L_\lambda(\mathbb{Z}, r) \otimes \mathbb{Z}_p \cong L_{\lambda'}(\mathbb{Z})/L_{\lambda'}(\mathbb{Z}, r) \otimes \mathbb{Z}_p$ because $\mathbb{Z}_p$ is a flat $\mathbb{Z}$-module. Thus, the map induced by $v_\lambda \mapsto v_{\lambda'}$ yields an isomorphism

$$L_\lambda(\mathbb{Z}_p)/L_\lambda(\mathbb{Z}_p, r) \cong L_{\lambda'}(\mathbb{Z}_p)/L_{\lambda'}(\mathbb{Z}_p, r) \tag{14}$$

of $\mathbb{Z}_p$-modules, if $\lambda, \lambda' \in \Gamma_{\mathrm{sc}}$ satisfy the conditions of (3.3) Proposition. Since the isomorphism (xc) commutes with the action of $\mathcal{S}$ and since $\mathcal{S}_p$ is generated by $\mathcal{S}$ and $\mathbb{Z}_p$ we see that (xc) is an isomorphism of $\mathcal{S}_p$-modules.
Let now $x_\alpha(t)$ be one of the generators of $K_*(p)$, i.e. $t \in \mathbb{Z}_p$ if α is negative and $t \in p^{\mathrm{ht}(\alpha)}\mathbb{Z}_p$ if α is positive. Since

$$\pi_{\lambda^\circ}(x_\alpha(t)) = \sum_{n \geq 0} \rho_\lambda(\frac{t^n x_\alpha^n}{n!}) \tag{15}$$

and $t^n \frac{x_\alpha^n}{n!}$ is contained in S_p we obtain that $L_\lambda(\mathbb{Z}_p, r)$ is $K_*(p)$-invariant. Moreover, since equation (15) holds with λ replaced by λ' equation (14) implies that

$$L_\lambda(\mathbb{Z}_p)/L_\lambda(\mathbb{Z}_p, r) \cong L_{\lambda'}(\mathbb{Z}_p)/L_{\lambda'}(\mathbb{Z}_p, r) \tag{16}$$

as $K_*(p)$-modules, if $\lambda, \lambda' \in \Gamma_{\mathrm{sc}}$ satisfy the conditions of (3.3) Proposition.

(4.4.2) We come to the situation of interest to us. We select a subgroup $\Gamma \leq \mathbf{G}(\mathbb{Z})$ which satisfies the following local condition at p:

$$\Gamma \leq K_*(p).$$

Since $L_\lambda(\mathbb{Z}_p, r)$ in particular is Γ-invariant we obtain

$$\Gamma L_\lambda(\mathbb{Z}, r) \leq L_\lambda(\mathbb{Z}) \cap L_\lambda(\mathbb{Z}_p, r) = L_\lambda(\mathbb{Z}, r).$$

Hence, we see that $L_\lambda(\mathbb{Z})/L_\lambda(\mathbb{Z}, r)$ is a Γ-module. Moreover, since the isomorphism $L_\lambda(\mathbb{Z})/L_\lambda(\mathbb{Z}, r) \cong L_{\lambda'}(\mathbb{Z})/L_{\lambda'}(\mathbb{Z}, r)$ is the restriction of the isomorphism (et) (both send v_λ to $v_{\lambda'}$) we deduce that $L_\lambda(\mathbb{Z})/L_\lambda(\mathbb{Z}, r) \cong L_{\lambda'}(\mathbb{Z})/L_{\lambda'}(\mathbb{Z}, r)$ as Γ-modules. Thus, we obtain the following

Proposition. *1.) Let $\lambda \in \Gamma_{\rm sc}$ be integral and dominant satisfying equation (13). The submodule $L_\lambda(\mathbb{Z}, r)$ of $L_\lambda(\mathbb{Z})$ is Γ-invariant, hence, $L_\lambda(\mathbb{Z})/L_\lambda(\mathbb{Z}, r)$ is a Γ-module.*
2.) Let $\lambda, \lambda' \in \Gamma_{\lambda_0}$ be integral and dominant weights satisfying equation (13) and

- $m_\alpha = \langle \lambda, \alpha \rangle \geq r$ *for all* $\alpha \in \Delta$ *and* $m'_\alpha = \langle \lambda', \alpha \rangle \geq r$ *for all* $\alpha \in \Delta$
- $\lambda \equiv \lambda' \pmod{p^{2r}\Gamma_{\rm ad}}$.

Then

$$L_\lambda(\mathbb{Z})/L_\lambda(\mathbb{Z}, r) \cong L_{\lambda'}(\mathbb{Z})/L_{\lambda'}(\mathbb{Z}, r)$$

as Γ-modules.

As above, we call the Γ-module $L_\lambda(\mathbb{Z})/L_\lambda(\mathbb{Z}, r)$ the Truncation of length r of the irreducible highest weight module $L_\lambda(\mathbb{Z})$.

5 The Hecke operator

(5.1) Irreducible representations of Reductive groups. We introduce the Hecke operator acting on cohomology of irreducible representations of algebraic groups. Since we want to define the full set of Hecke operators, from now on we consider reductive groups. Thus, we let $\tilde{\mathbf{G}}/\mathbb{Q}$ be a connected reductive algebraic group, which splits over $\mathbb{Q}$. We denote by $\tilde{\mathbf{T}}$ a maximal $\mathbb{Q}$-split torus in $\tilde{\mathbf{G}}$ and we denote by $\mathbf{G} = (\tilde{\mathbf{G}}, \tilde{\mathbf{G}})$ the derived group. $\mathbf{G}$ is a semi simple $\mathbb{Q}$-split group and $\tilde{\mathbf{T}}$ contains a maximal $\mathbb{Q}$-split torus $\mathbf{T}$ of $\mathbf{G}$ (cf. [S], Prop. 8.1.8 (iii), p. 135). We denote by $\mathfrak{g}$ the Lie algebra of $\mathbf{G}$, hence, $\mathbf{G} = \mathbf{G}_{\rho_0}$ for some faithful representation $(\rho_{\lambda_0}, L_{\lambda_0})$ of $\mathfrak{g}$ of highest weight $\lambda_0 \in \Gamma_{\rm sc}$. We choose the Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ and the basis Δ of the root system of $(\mathfrak{g}, \mathfrak{h})$ such that $\mathbf{T}$ is given as in (4.3). We know that

$$\tilde{\mathbf{G}}(\bar{\mathbb{Q}}) = \mathbf{G}(\bar{\mathbb{Q}}) \times_\mu \mathbb{G}_m^a(\bar{\mathbb{Q}}),$$

where $\mathbb{G}_m^a(\bar{\mathbb{Q}}) = {\rm Rad}(\tilde{\mathbf{G}}(\bar{\mathbb{Q}}))$ is the radical of $\tilde{\mathbf{G}}(\bar{\mathbb{Q}})$ and $\mu \leq \mathbf{G}(\bar{\mathbb{Q}}) \cap \mathbb{G}_m^a(\bar{\mathbb{Q}})$ is a finite subgroup. We note that $\mathbb{G}_m^a(\bar{\mathbb{Q}})$ is contained in the centre of $\tilde{\mathbf{G}}(\bar{\mathbb{Q}})$, hence, μ is contained in the centre of $\mathbf{G}(\bar{\mathbb{Q}})$ (cf. [S], Prop. 7.3.1, p. 120 and Cor. 8.1.6, p. 134). In particular, we obtain

$$\tilde{\mathbf{T}}(\bar{\mathbb{Q}}) = \mathbf{T}(\bar{\mathbb{Q}}) \times_\mu \mathbb{G}_m^a(\bar{\mathbb{Q}}),$$

i.e. any $t \in \tilde{\mathbf{T}}(\bar{\mathbb{Q}})$ can be written

$$t = t^0 z, \tag{1}$$

where $t^0 \in \mathbf{T}(\bar{\mathbb{Q}})$ and $z \in \mathbb{G}_m^a(\bar{\mathbb{Q}})$. The character group of $\tilde{\mathbf{T}}$ is given as

$$X(\tilde{\mathbf{T}}) = \{\lambda^\circ \otimes \kappa,\ \lambda^\circ \in X(\mathbf{T}), \kappa \in X(\mathbb{G}_m^a):\ \lambda^\circ|_\mu = \kappa|_\mu\};$$

explicitly, the character $\lambda^\circ \otimes \kappa$ is defined on $\bar{\mathbb{Q}}$-points as

$$\lambda^\circ \otimes \kappa(t^0 z) := \lambda^\circ(t^0)\kappa(z), \tag{2}$$

where $t^0 \in \mathbf{T}(\bar{\mathbb{Q}})$ and $z \in \mathbb{G}_m^a(\bar{\mathbb{Q}})$. In particular, since the simple roots $\alpha^\circ \in X(\mathbf{T})$, $\alpha \in \Delta$ (cf. equation (5) in section (4.3)), vanish on the centre of $\mathbf{G}(\bar{\mathbb{Q}})$, hence, on μ, they extend to characters of $\tilde{\mathbf{T}}(\bar{\mathbb{Q}})$ by defining $\alpha^\circ(t^0 z) = \alpha^\circ(t^0)$, $t^0 \in \mathbf{T}(\bar{\mathbb{Q}})$, $z \in \mathbb{G}_m^a(\bar{\mathbb{Q}})$.

We fix an integral and dominant weight $\tilde{\lambda} \in X(\tilde{\mathbf{T}})$. We assume that $\tilde{\lambda}$ decomposes

$$\tilde{\lambda} = \lambda^\circ \otimes \kappa, \tag{3}$$

where $\kappa = \tilde{\lambda}|_{\mathbb{G}_m^a}$ and $\lambda^\circ = \tilde{\lambda}|_{\mathbf{T}} \in X(\mathbf{T})$ is the image of an integral and dominant weight $\lambda \in \Gamma_{\lambda_0}$ under the map $\lambda \mapsto \lambda^\circ$ (cf. equation (6) in section (4.3)). We denote by

$$\pi_{\tilde{\lambda}} : \tilde{\mathbf{G}} \to \mathbf{GL}(L_{\tilde{\lambda}}(\mathbb{Z})) = \mathbf{GL}_m$$

the irreducible representation of the algebraic group $\tilde{\mathbf{G}}$ of highest weight $\tilde{\lambda}$ (as in section (4.3) we identify the algebraic groups $\mathbf{GL}(L_{\tilde{\lambda}}(\mathbb{Z})) = \mathbf{GL}_m/\mathbb{Z}$, where $m = \dim L_{\tilde{\lambda}}(\mathbb{Z})$). The representation $\pi_{\tilde{\lambda}}$ is defined over $\mathbb{Z}$; it is given as

$$\pi_{\tilde{\lambda}} = \pi_{\lambda^\circ} \otimes \kappa,$$

where λ° and κ are as in equation (3) and $\pi_{\lambda^\circ} : \mathbf{G}/\mathbb{Z} \to \mathbf{SL}(L_\lambda(\mathbb{Z}))$ is the irreducible representation of $\mathbf{G}$ of highest weight λ° (cf. equation (10) in section (4.3); note that $\lambda \in \Gamma_{\lambda_0}$, hence, $\Gamma_{\lambda_0} \supseteq \Gamma_\lambda$). Explicitly, this means

$$\pi_{\tilde{\lambda}}(g) = \kappa(z)\, \pi_{\lambda^\circ}(g^0) \tag{4}$$

for any $g = g^0 z \in \tilde{\mathbf{G}}(\bar{\mathbb{Q}})$ with $g^0 \in \mathbf{G}(\bar{\mathbb{Q}})$ and $z \in \mathbb{G}_m^a(\bar{\mathbb{Q}})$. The representation $\pi_{\tilde{\lambda}}$ is well defined because λ° and κ coincide on μ.

We note that the representation space of $\pi_{\tilde{\lambda}}$ is the same as the representation space of π_{λ°, i.e. $L_{\tilde{\lambda}}(R) = L_\lambda(R)$ as $\mathbb{Z}$-modules for any $\mathbb{Z}$-algebra R. Hence, any subspace of $L_\lambda(R)$ also is a subspace of $L_{\tilde{\lambda}}(R)$. In particular, $L_\lambda(\mathbb{Z}, r)$ is a subspace of $L_{\tilde{\lambda}}(\mathbb{Z})$ and if we want to indicate that we view $L_\lambda(\mathbb{Z}, r)$ as a subspace of $L_{\tilde{\lambda}}(\mathbb{Z})$ we write it as $L_{\tilde{\lambda}}(\mathbb{Z}, r)$. Similarly, if we want to indicate that we view the weight space $L_\lambda(R, \mu) \le L_\lambda(R)$, $\mu \in \Pi_\lambda$, as a subspace of $L_{\tilde{\lambda}}(R)$ we write it as $L_{\tilde{\lambda}}(R, \mu)$. Thus, in contrast to $L_\lambda(\bar{\mathbb{Q}}, \mu)$, on $L_{\tilde{\lambda}}(\bar{\mathbb{Q}}, \mu)$ we have an action of the torus $\tilde{\mathbf{T}}(\bar{\mathbb{Q}})$; explicitly, since $\mathbf{T}(\bar{\mathbb{Q}})$ acts on $L_\lambda(\bar{\mathbb{Q}}, \mu)$ via μ° (cf. equation (7) in section (4.3)) we find for any $t = t^0 z \in \tilde{\mathbf{T}}(\bar{\mathbb{Q}})$ as in equation (1) and any $v \in L_{\tilde{\lambda}}(\bar{\mathbb{Q}}, \mu)$ $(= L_\lambda(\bar{\mathbb{Q}}, \mu))$

$$\pi_{\tilde{\lambda}}(t)v = \kappa(z)\pi_{\lambda^\circ}(t^0)v = \kappa(z)\mu^\circ(t^0)v = (\mu^\circ \otimes \kappa)(t)v. \tag{5}$$

Thus, $\tilde{\mathbf{T}}(\bar{\mathbb{Q}})$ acts on $L_{\tilde{\lambda}}(\bar{\mathbb{Q}}, \mu)$ via the character $\mu^0 \otimes \kappa$. In particular,

$$L_{\tilde{\lambda}}(\mathbb{Z}) = \bigoplus_{\mu \in \Pi_\lambda} L_{\tilde{\lambda}}(\mathbb{Z}, \mu) \tag{6}$$

is the weight decomposition of $L_{\tilde{\lambda}}(\mathbb{Z})$ with respect to the torus $\tilde{\mathbf{T}}$.

(5.2) The Hecke operator acting on cohomology. In the remainder of section 5, we fix a $\mathbb{Q}$-split reductive group $\tilde{\mathbf{G}}$. We use the notations introduced in section (5.1), e.g. $\mathbf{G} = (\tilde{\mathbf{G}}, \tilde{\mathbf{G}})$ is the derived group of $\tilde{\mathbf{G}}$ and $\mathbf{G} = \mathbf{G}_{\lambda_0}$ for some irreducible representation $(\rho_{\lambda_0}, L_{\lambda_0})$ of the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$. We recall that $\mathbf{G}$ has a natural $\mathbb{Z}$-structure $\mathbf{G}/\mathbb{Z} = \mathbf{G}_{\lambda_0}/\mathbb{Z}$. Moreover, we fix a prime $p \in \mathbb{N}$ and a congruence subgroup $\Gamma \leq \mathbf{G}(\mathbb{Z})$ such that Γ satisfies the following local condition at p:

$$\Gamma \leq K_*(p).$$

We denote by $\tilde{\lambda} \in X(\tilde{\mathbf{T}})$ a dominant and integral weight and we write $\tilde{\lambda}$ as

$$\tilde{\lambda} = \lambda^\circ \otimes \kappa$$

where λ° and κ are as in equation (3). The highest weight module $(\pi_{\tilde{\lambda}}, L_{\tilde{\lambda}}(\mathbb{Z}))$ of $\tilde{\mathbf{G}}$ decomposes $\pi_{\tilde{\lambda}} = \pi_{\lambda^\circ} \otimes \kappa$ (cf. equation (4)). Since $\pi_{\tilde{\lambda}}$ is defined over $\mathbb{Z}$, the group $\Gamma \leq \mathbf{G}(\mathbb{Z})$ leaves the lattice $L_{\tilde{\lambda}}(\mathbb{Z}) = L_\lambda(\mathbb{Z})$ invariant, i.e. $L_{\tilde{\lambda}}(\mathbb{Z})$ is a Γ-module and the cohomology groups

$$H^i(\Gamma, L_{\tilde{\lambda}}(R))$$

are defined for any $\mathbb{Z}$-algebra R. In this section, using the $\tilde{\mathbf{T}}(\mathbb{Q})$-module structure on $L_{\tilde{\lambda}}(\mathbb{Q})$ we will define Hecke operators acting on $H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}))$ and we will examine the action of the Hecke operator on the cohomology of truncated modules. Together with the properties of the truncation of highest weight modules and some cohomological formalism this will yield our main result in section 6: a bound for the dimension of the slope subspaces of $H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Q}))$, which is independent of the weight $\tilde{\lambda}$.

We fix strictly positive integers $e_\alpha \in \mathbb{N}$, $\alpha \in \Delta$, and an element in $h \in \tilde{\mathbf{T}}(\mathbb{Q})$, which satisfies

$$\alpha^\circ(h) = p^{e_\alpha} \tag{7}$$

for all $\alpha \in \Delta$. We define the Hecke operator

$$T(h) = \Gamma h \Gamma.$$

The group Γ and the torus $\tilde{\mathbf{T}}(\mathbb{Q})$ act on $L_{\tilde{\lambda}}(\mathbb{Q})$ and this induces an action of the Hecke operator $T(h)$ on the space of cochains $C^h(\Gamma, L_{\tilde{\lambda}}(\mathbb{Q}))$. We recall the definition of this action. We decompose the double coset

$$T(h) = \bigcup_{i=1,\dots,d} \Gamma h \gamma_i,$$

where the γ_i run over a (fixed) system of representatives for $h^{-1}\Gamma h \cap \Gamma \backslash \Gamma$. For any $\eta \in \Gamma$ and any index i satisfying $1 \le i \le d$ there is an index $\eta(i)$ such that

$$\Gamma h \gamma_i \eta = \Gamma h \gamma_{\eta(i)},$$

In particular, there are $\rho_i(\eta) \in \Gamma$, $i = 1, \ldots, d$, such that $h\gamma_i\eta = \rho_i(\eta) h \gamma_{\eta(i)}$. Let now $c \in C^h(\Gamma, L_{\tilde{\lambda}}(\mathbb{Q}))$ be any cochain; we then define $T(h)(c)$ as the cochain $c' \in C^h(\Gamma, L_{\tilde{\lambda}}(\mathbb{Q}))$, which is given by

$$c'(\eta_0, \ldots, \eta_h) = \sum_{1 \le i \le d} (h\gamma_i)^{-1} c(\rho_i(\eta_0), \ldots, \rho_i(\eta_h) \tag{8}$$

(cf. [K-P-S], p. 227). Thus, $T(h)$ defines an operator on $C^\bullet(\Gamma, L_{\tilde{\lambda}}(\mathbb{Q}))$. Since the action of $T(h)$ commutes with the coboundary operator, $T(h)$ acts on cohomology with coefficients in $L_{\tilde{\lambda}}(\mathbb{Q})$, i.e. $T(h)$ defines an element in $\mathrm{End}(H^\bullet(\Gamma, L_{\tilde{\lambda}}(\mathbb{Q})))$.

We note that the Hecke algebra acts from the right on cohomology but since we only consider a single Hecke operator T this will not become relevant and we therefore write the image of a cohomology class c under T as $T(c)$.

(5.3) Normalization of Hecke operators acting on cohomology. In general, the action of $T(h)$ on cochains does not leave the lattice $C^h(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}))$ in $C^h(\Gamma, L_{\tilde{\lambda}}(\mathbb{Q}))$ invariant. To achieve this, we have to normalize the Hecke operator as follows. For any $\tilde{\lambda} \in X(\tilde{\mathbf{T}})$ we define the normalized operator $\mathbb{T}(h)$ as

$$\mathbb{T}(h) = \tilde{\lambda}(h)\, T(h) \ \in \mathrm{End}(C^\bullet(\Gamma, L_{\tilde{\lambda}}(\mathbb{Q}))). \tag{9}$$

We note that in the rank-1 case the normalized Hecke operator $\mathbb{T}$ corresponds to the classical Hecke operator on modular forms.

(5.4) Lemma. *Let $\tilde{\lambda} \in X(\tilde{\mathbf{T}})$ be an integral and dominant weight. The Hecke operator $\mathbb{T}(h)$ leaves $C^\bullet(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}))$ and, hence, $C^\bullet(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}) \otimes \mathbb{Z}/(p^r))$ invariant.*

Proof. We write $\tilde{\lambda} = \lambda^\circ \otimes \kappa$ as in equation (3) and we write the element defining the Hecke operator as $h = h^0 z$, $h^0 \in \mathbf{T}(\bar{\mathbb{Q}})$, $z \in \mathbb{G}_m^a(\bar{\mathbb{Q}})$. Equation (7) implies that

$$\alpha^\circ(h^0) = p^{e_\alpha} \tag{10}$$

for all simple roots $\alpha \in \Delta$. We let $v_\mu \in L_{\tilde{\lambda}}(\mathbb{Z}, \mu)$, $\mu \in \Pi_\lambda$, be arbitrary (cf. equation (6)). Equation (5) implies that v_μ has weight $\mu^\circ \otimes \kappa$, hence, we obtain

$$\pi_{\tilde{\lambda}}(h) v_\mu = \kappa(z) \mu^\circ(h^0) v_\mu.$$

Since $\mu \in \Pi_\lambda$, it has the form $\mu = \lambda - \sum_{\alpha \in \Delta} c_\alpha \alpha$ with all $c_\alpha \in \mathbb{N}_0$ and we further obtain using equation (7)

$$\begin{aligned} \kappa(z)\mu^\circ(h^0) &= \kappa(z)\lambda^\circ(h^0) \prod_{\alpha \in \Delta} (\alpha^\circ)^{-c_\alpha}(h^0) \\ &= \tilde{\lambda}(h) \prod_{\alpha \in \Delta} (\alpha^\circ)^{-c_\alpha}(h^0) \\ &= \tilde{\lambda}(h) \prod_{\alpha \in \Delta} p^{-e_\alpha c_\alpha}. \end{aligned}$$

Thus, we obtain

$$\tilde{\lambda}(h)\pi_{\tilde{\lambda}}(h^{-1})v_\mu = \prod_{\alpha\in\Delta} p^{e_\alpha c_\alpha} v_\mu.$$

Since e_α is strictly positive (cf. equation (7)) we deduce that

$$\sum_{\alpha\in\Delta} c_\alpha e_\alpha \geq \sum_{\alpha\in\Delta} c_\alpha = \mathrm{ht}_\lambda(\mu).$$

Hence, we obtain: if $v_\mu \in L_{\tilde{\lambda}}(\mathbb{Z},\mu)$ then $\tilde{\lambda}(h)\pi_{\tilde{\lambda}}(h^{-1})v_\mu = c_\mu v_\mu$ with

$$c_\mu \in p^{\mathrm{ht}_\lambda(\mu)}\mathbb{Z}. \tag{11}$$

We let $c \in C^h(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}))$ be arbitrary and we fix a tuple $(\eta_0,\ldots,\eta_h) \in \Gamma^{h+1}$. We write

$$c(\rho_i(\eta_0),\ldots,\rho_i(\eta_h)) = \sum_\mu v_\mu^i, \tag{12}$$

where $v_\mu^i \in L_{\tilde{\lambda}}(\mathbb{Z},\mu)$ and μ runs over all weights in Π_λ. From the definition of the action of the Hecke operator we obtain

$$\begin{aligned} (\mathbb{T}(h)c)(\eta_0,\ldots,\eta_h) &= \tilde{\lambda}(h)\sum_i \gamma_i^{-1}h^{-1}c(\rho_i(\eta_0),\ldots,\rho_i(\eta_h)) \\ &= \sum_i\sum_\mu \gamma_i^{-1}\tilde{\lambda}(h)h^{-1}v_\mu^i. \end{aligned} \tag{13}$$

Since $\mathrm{ht}_\lambda(\mu) \geq 0$, equation (11) implies that $\tilde{\lambda}(h)h^{-1}v_\mu^i$ is contained in $L_{\tilde{\lambda}}(\mathbb{Z},\mu)$ and since $\gamma_i \in \Gamma$ leaves $L_{\tilde{\lambda}}(\mathbb{Z}) = L_\lambda(\mathbb{Z})$ invariant, equation (13) implies

$$(\mathbb{T}(h)(c))(\eta_0,\ldots,\eta_h) \in L_{\tilde{\lambda}}(\mathbb{Z}).$$

Since $(\eta_0,\ldots,\eta_h)$ was arbitrary this implies that $\mathbb{T}(h)(c)$ is contained in $C^h(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}))$, which proves the first claim. Since the Hecke operator $\mathbb{T}(h)$ leaves the subspace $C^\bullet(\Gamma, p^r L_{\tilde{\lambda}}(\mathbb{Z})) = p^r C^\bullet(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}))$ in $C^\bullet(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}))$ invariant, $\mathbb{T}(h)$ also acts on $C^\bullet(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z})/p^r L_{\tilde{\lambda}}(\mathbb{Z}))$, which implies the second claim. Thus, the proof of the Lemma is complete.

Remark. The Lemma implies that $\mathbb{T}(h)$ defines an operator on $H^\bullet(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}))$ and on $H^\bullet(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}/(p^r)))$.

(5.5) Cohomology of truncating modules. We denote by $L_{\tilde{\lambda}}(\mathbb{Z}/(p^r), r)$ the image of our truncating module $L_{\tilde{\lambda}}(\mathbb{Z}, r) \leq L_{\tilde{\lambda}}(\mathbb{Z})$ under the canonical map

$$L_{\tilde{\lambda}}(\mathbb{Z}) \to L_{\tilde{\lambda}}(\mathbb{Z}/(p^r)) = L_{\tilde{\lambda}}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/(p^r), \tag{14}$$

which sends $v \mapsto v \otimes 1$. Thus, $L_{\tilde{\lambda}}(\mathbb{Z}/(p^r), r)$ is a submodule of $L_{\tilde{\lambda}}(\mathbb{Z}/(p^r))$ and

$$L_{\tilde{\lambda}}(\mathbb{Z}/(p^r), r) = \bigoplus_{\substack{\mu\in\Pi_\lambda \\ \mathrm{ht}_\lambda(\mu)\leq r}} p^{r-\mathrm{ht}_\lambda(\mu)} L_{\tilde{\lambda}}(\mathbb{Z},\mu) \oplus \bigoplus_{\substack{\mu\in\Pi_\lambda \\ \mathrm{ht}_\lambda(\mu)> r}} L_{\tilde{\lambda}}(\mathbb{Z},\mu) \pmod{p^r L_{\tilde{\lambda}}(\mathbb{Z})}.$$

Lemma. *1.)* $\mathbb{T}(h)$ *leaves the submodules* $C^\bullet(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}, r)) \leq C^\bullet(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}))$ *and* $C^\bullet(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}/(p^r), r)) \leq C^\bullet(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}/(p^r))$ *invariant.*
2.) $\mathbb{T}(h)$ *annihilates* $C^\bullet(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}/(p^r), r))$.

We note that (5.4) Lemma implies that $\mathbb{T}(h)$ acts on $C^\bullet(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}))$ and on $C^\bullet(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}/(p^r))$.

Proof. We prove 1.) and 2.) together. We let $c \in C^h(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}, r))$ and we fix a tuple $(\eta_0, \dots, \eta_h) \in \Gamma^{h+1}$. We write

$$c(\rho_i(\eta_0), \dots, \rho_i(\eta_h)) = \sum_\mu v_\mu^i,$$

where μ runs over all weights in Π_λ and

$$v_\mu^i \in \begin{cases} p^{r-\mathrm{ht}_\lambda(\mu)} L_{\tilde{\lambda}}(\mathbb{Z}, \mu) & \text{if} \quad \mathrm{ht}_\lambda(\mu) \leq r \\ L_{\tilde{\lambda}}(\mathbb{Z}, \mu) & \text{if} \quad \mathrm{ht}_\lambda(\mu) > r. \end{cases} \tag{15}$$

Again, using equation (11) we find

$$\tilde{\lambda}(h) h^{-1} v_\mu^i = c_{\mu,i} v_\mu^i, \tag{16}$$

where $c_{\mu,i} \in p^{\mathrm{ht}_\lambda(\mu)} \mathbb{Z}$. Hence, equation (15) implies that

$$\tilde{\lambda}(h) h^{-1} v_\mu^i \in p^r L_{\tilde{\lambda}}(\mathbb{Z}).$$

Taking into account that $\gamma_i \in \Gamma$ leaves $p^r L_{\tilde{\lambda}}(\mathbb{Z})$ invariant, the definition of the action of the Hecke operator (cf. equation (13)) implies

$$\mathbb{T}(h)(c)(\eta_0, \dots, \eta_h) \in p^r L_{\tilde{\lambda}}(\mathbb{Z}).$$

Since $p^r L_{\tilde{\lambda}}(\mathbb{Z}) \subseteq L_{\tilde{\lambda}}(\mathbb{Z}, r)$ this implies that $\mathbb{T}(h)(c)$ is contained in $C^h(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}, r))$, hence, $\mathbb{T}(h)$ leaves $C^h(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}, r))$ invariant. Moreover, it implies that $\mathbb{T}(h)$ annihilates the submodule $C^h(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}/(p^r), r))$ of $C^h(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}/(p^r)))$. Thus, the proof of the Lemma is complete.

(5.6) Remark. The Lemma in particular implies that $\mathbb{T}(h)$ acts on $H^\bullet(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}, r))$ and on $H^\bullet(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}/(p^r), r))$ and that $\mathbb{T}(h)$ annihilates $H^\bullet(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}/(p^r), r))$. Moreover, (5.4) Lemma and (5.5) Lemma imply that $\mathbb{T}(h)$ acts on $C^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z})/L_{\tilde{\lambda}}(\mathbb{Z}, r)) \cong C^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}))/C^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}, r))$, hence $\mathbb{T}(h)$ defines an operator on $H^\bullet(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z})/L_{\tilde{\lambda}}(\mathbb{Z}, r))$.

6 Boundedness of slope spaces

(6.1) The slope subspace. We fix a finite dimensional $\mathbb{Q}$-vector space V on which the operator T acts and we assume that T leaves a $\mathbb{Z}$-lattice $V_\mathbb{Z}$ in V invariant. We select an $\alpha \in \mathbb{Q}_{\geq 0}$ and we define the slope α subspaces $V^\alpha \leq V$ and $V_\mathbb{Z}^\alpha \leq V_\mathbb{Z}$ as follows. We

denote by $m(X) \in \mathbb{Q}[X]$ the characteristic polynomial of T acting on V. Since T leaves $V_{\mathbb{Z}}$ invariant we know that $m(X) \in \mathbb{Z}[X]$. We set

$$m_\alpha(X) = \prod_{\mu,\ v_p(\mu)\neq\alpha} (X-\mu) \in \mathbb{Z}[X],$$

where μ runs over all roots of $m(X)$ in a splitting field for T (counted with their multiplicities), which have p-adic value not equal to α. We note that $m_\alpha(X)$ is contained in $\mathbb{Z}[X]$. We then define the slope α-subspaces V^α resp. $V^\alpha_{\mathbb{Z}}$ as $V^\alpha = m_\alpha(T)V$ resp. as $V^\alpha_{\mathbb{Z}} = V^\alpha \cap V_{\mathbb{Z}}$. We note that $V^\alpha_{\mathbb{Z}}$ is a lattice in V^α. More generally, we let K be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$. We view K as a subfield of $\bar{\mathbb{Q}}_p$, hence, v_p induces a p-adic valuation on K. We set $V_K = V \otimes_{\mathbb{Q}} K$ and $V_{\mathcal{O}} = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}$ and as above we define the slope subspaces $V^\alpha_K = m_\alpha(T)V_K$ and $V^\alpha_{\mathcal{O}} = V_{\mathcal{O}} \cap V^\alpha_K$. Again, $V^\alpha_{\mathcal{O}}$ is a lattice in V^α_K. There are inclusions

$$\bigoplus_{\alpha\geq 0} m_\alpha(T)V_{\mathcal{O}} \subseteq \bigoplus_{\alpha\geq 0} V^\alpha_{\mathcal{O}} \subseteq V_{\mathcal{O}},$$

which in general are strict inclusions, i.e. equality does not hold in general in the above inclusions. We set

$$d(\alpha) = \mathrm{rk}_{\mathbb{Z}} V^\alpha_{\mathbb{Z}} = \dim_{\mathbb{Q}} V^\alpha.$$

Since the slope decomposition is defined over $\mathbb{Q}$ we know that $d(\alpha) = \dim_K V^\alpha_K = \mathrm{rk}_{\mathcal{O}} V^\alpha_{\mathcal{O}}$.

(6.2) A general estimate on the rank of slope subspaces. We keep the notations from section (6.1). In addition, we select a splitting field by $K/\mathbb{Q}_p$ for T acting on V; thus, $V^\alpha_K = \bigoplus_\mu V_K(\mu)$ is the sum of the generalized eigenspaces $V_K(\mu)$, which are attached to eigenvalues $\mu \in \mathcal{O}$ of T satisfying $v_p(\mu) = \alpha$. We denote by $\mathfrak{p}$ the maximal ideal in $\mathcal{O}$, i.e. $\mathfrak{p} = \{x \in \mathcal{O} : v_p(x) > 0\}$, e resp. f is the ramification index resp. the inertia degree of $\mathfrak{p}|p$, $\varpi \in \mathcal{O}$ is a local prime and $v_{\mathfrak{p}}$ is the $\mathfrak{p}$-adic valuation attached to $\mathfrak{p}$. We note that $V^\alpha \neq 0$ implies that $\alpha \in \frac{1}{e}\mathbb{N}_0$. Thus, if α is a non-trivial slope we know that $e\alpha \in \mathbb{N}_0$ is an integer. Moreover, since T leaves $V^\alpha_{\mathcal{O}}$ invariant, it induces an operator on $V^\alpha_{\mathcal{O}} \otimes \mathcal{O}/\mathfrak{p}^r$.

(6.2.1) Lemma. *a.) Let $\alpha \in \mathbb{Q}_{\geq 0}$ be a non-trivial slope for V (i.e. $V^\alpha \neq 0$) and let r be any integer satisfying $r > e\alpha$. Then*

$$p^{d(\alpha)f(r-e\alpha)} \,\Big|\, \left|\frac{V^\alpha_{\mathcal{O}} \otimes \mathcal{O}/\mathfrak{p}^r}{\ker T|_{V^\alpha_{\mathcal{O}} \otimes \mathcal{O}/\mathfrak{p}^r}}\right|.$$

b.) If r in addition is sufficiently large with largeness only depending on T, then

$$p^{d(\alpha)f(r-e\alpha)} = \left|\frac{V^\alpha_{\mathcal{O}} \otimes \mathcal{O}/\mathfrak{p}^r}{\ker T|_{V^\alpha_{\mathcal{O}} \otimes \mathcal{O}/\mathfrak{p}^r}}\right|.$$

Proof. a.) We abbreviate $q = p^f$ and $d = d(\alpha)$. Since T is split over K there is a basis $\mathcal{B}$ of V^α_K such that $T|_{V^\alpha_K}$ is represented by a triangular matrix

$$\mathcal{D}_{\mathcal{B}}(T|_{V^\alpha_K}) = \begin{pmatrix} \mu_1 & & * \\ & \ddots & \\ & & \mu_d \end{pmatrix}.$$

Here, $\mu_1, \ldots, \mu_d \in \mathcal{O}$ are the eigenvalues of T acting on V_K^α counted with their respective multiplicities; hence, $v_p(\mu_i) = \alpha$ for all $i = 1, \ldots, d$ and since $\det T|_{V_K^\alpha} = \prod_i \mu_i$ we obtain

$$(1) \qquad v_p(\det T|_{V_K^\alpha}) = \sum_{i=1}^{d} v_p(\mu_i) = d\alpha.$$

On the other hand, since $V_{\mathcal{O}}^\alpha$ is torsionfree it is a free $\mathcal{O}$-module and we choose a basis $\mathcal{C}$ of $V_{\mathcal{O}}^\alpha$. Thus, the representing matrix $\mathcal{D}_{\mathcal{C}}(T|_{V_K^\alpha})$ has coefficients in $\mathcal{O}$ and the Theorem on elementary divisors implies that there are matrices $A, B \in \mathrm{GL}_d(\mathcal{O})$ such that

$$(2) \qquad A\mathcal{D}_{\mathcal{C}}(T|_{V_K^\alpha})B = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{pmatrix}$$

is diagonal. Here, $\lambda_1, \ldots, \lambda_d \in \mathcal{O}$ are the elementary divisors of the representing matrix $\mathcal{D}_{\mathcal{C}}(T|_{V_K^\alpha})$. Since $\det A, \det B \in \mathcal{O}^*$, equation (2) implies that $v_p(\det T) = \sum_i v_p(\lambda_i)$, hence, using equation (1) we obtain

$$(3) \qquad \sum_{i=1}^{d} v_p(\lambda_i) = d\alpha.$$

There are basis $(c_i)_i$ and $(d_i)_i$ of $V_{\mathcal{O}}^\alpha$ such that equation (2) gives the representing matrix of $T|_{V_K^\alpha}$ with respect to the pair of basis $(c_i)_i$ and $(d_i)_i$. Let now $v = \sum_i \beta_i c_i \in V_{\mathcal{O}}^\alpha$ be arbitrary. Equation (2) implies that $T(v) = \sum_{i=1}^d \lambda_i \beta_i d_i$, hence, $T(v)$ is contained in $\mathfrak{p}^r V_{\mathcal{O}}^\alpha$ precisely if $\beta_i \in \mathfrak{p}^{r - v_{\mathfrak{p}}(\lambda_i)}$ in case $v_{\mathfrak{p}}(\lambda_i) \leq r$ and $\beta_i \in \mathcal{O}$ in case $v_{\mathfrak{p}}(\lambda_i) > r$. We obtain

$$(4) \qquad |\ker T|_{V_{\mathcal{O}}^\alpha \otimes \mathcal{O}/\mathfrak{p}^r}| = \prod_{i=1}^{d} q^{\min\{r, v_{\mathfrak{p}}(\lambda_i)\}} = q^{\sum_i \min\{r, v_{\mathfrak{p}}(\lambda_i)\}}.$$

Consequently, $|\ker T|_{V_{\mathcal{O}}^\alpha \otimes \mathcal{O}/\mathfrak{p}^r}|$ divides $q^{\sum_i v_{\mathfrak{p}}(\lambda_i)}$. Using equation (3) and taking into account that $v_{\mathfrak{p}} = e v_p$ we thus obtain

$$|\ker T|_{V_{\mathcal{O}}^\alpha \otimes \mathcal{O}/\mathfrak{p}^r}| \,\big|\, q^{de\alpha}.$$

Since $|V_{\mathcal{O}}^\alpha \otimes \mathcal{O}/\mathfrak{p}^r| = q^{dr}$ this yields the claim.
b.) We note that $\lambda_1, \ldots, \lambda_d$ only depend on T. Hence, if we choose r such that $r > v_{\mathfrak{p}}(\lambda_i)$ for all $i = 1, \ldots, d$ we obtain equation (4) in the form

$$|\ker T|_{V_{\mathcal{O}}^\alpha \otimes \mathcal{O}/\mathfrak{p}^r}| = \prod_{i=1}^{d} q^{v_{\mathfrak{p}}(\lambda_i)} = q^{\sum_i v_{\mathfrak{p}}(\lambda_i)} = q^{de\alpha}.$$

Together with $|V_{\mathcal{O}}^\alpha \otimes \mathcal{O}/\mathfrak{p}^r| = q^{dr}$ this yields claim. Thus, the Lemma is proven.

Lemma (6.2.1) descends to $\mathbb{Q}$:

(6.2.2) Proposition. *a.) Let $\alpha \in \mathbb{Q}_{\geq 0}$ be a non-trivial slope for V and let r be any integer satisfying $r > \alpha$. Then*

$$p^{d(\alpha)(r-\alpha)} \Big| \left| \frac{V_{\mathbb{Z}}^{\alpha} \otimes \mathbb{Z}/(p^r)}{\ker T|_{V_{\mathbb{Z}}^{\alpha} \otimes \mathbb{Z}/(p^r)}} \right|.$$

b.) If in addition r is sufficiently large with largeness only depending on T, then

$$p^{d(\alpha)(r-\alpha)} = \left| \frac{V_{\mathbb{Z}}^{\alpha} \otimes \mathbb{Z}/(p^r)}{\ker T|_{V_{\mathbb{Z}}^{\alpha} \otimes \mathbb{Z}/(p^r)}} \right|.$$

Proof. We deduce the Proposition from Lemma (6.2.1) by extending scalars from $\mathbb{Q}$ to the splitting field $K/\mathbb{Q}_p$ of T. We select basis $(v_i)_i$ and $(w_i)_i$ of $V_{\mathbb{Z}}^{\alpha}$ such that $T|_{V_{\mathbb{Z}}^{\alpha}}$ is represented by a diagonal matrix $\mathcal{D} = \mathrm{diag}(\lambda_1, \ldots, \lambda_d)$, $\lambda_i \in \mathbb{Z}$. Hence, $T|_{V_{\mathbb{Z}}^{\alpha} \otimes \mathbb{Z}/(p^r)}$ is represented by the matrix $\bar{\mathcal{D}} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_d)$, $\bar{\lambda}_i = \lambda_i \pmod{p^r}$ and we see that

$$\ker T|_{V_{\mathbb{Z}}^{\alpha} \otimes \mathbb{Z}/(p^r)} = \bigoplus_{i=1}^{d} p^{\max(0, r - v_p(\lambda_i))} \mathbb{Z} \bar{v}_i \qquad (\bar{v}_i = v_i + p^r V_{\mathbb{Z}}^{\alpha}).$$

In particular, $|\ker T|_{V_{\mathbb{Z}}^{\alpha} \otimes \mathbb{Z}/(p^r)}| = \prod_i p^{\min(r, v_p(\lambda_i))}$. Quite analogous, we find that

$$\ker T|_{V_{\mathcal{O}}^{\alpha} \otimes \mathcal{O}/(p^r)} = \bigoplus_{i=1}^{d} \varpi^{\max(0, er - v_{\mathfrak{p}}(\lambda_i))} \mathcal{O} \bar{v}_i,$$

hence, $|\ker T|_{V_{\mathcal{O}}^{\alpha} \otimes \mathcal{O}/(p^r)}| = \prod_i q^{\min(er, v_{\mathfrak{p}}(\lambda_i))} = \prod_i p^{fe\min(r, v_p(\lambda_i))}$. Thus,

$$|\ker T|_{V_{\mathbb{Z}}^{\alpha} \otimes \mathbb{Z}/(p^r)}|^{ef} = |\ker T|_{V_{\mathcal{O}}^{\alpha} \otimes \mathcal{O}/(p^r)}|.$$

Since $|V_{\mathcal{O}}^{\alpha} \otimes \mathcal{O}/(p)^r| = |V_{\mathbb{Z}}^{\alpha} \otimes \mathbb{Z}/(p^r)|^{ef}$ we obtain

$$\left| \frac{V_{\mathcal{O}}^{\alpha} \otimes \mathcal{O}/(p)^r}{\ker T|_{V_{\mathcal{O}}^{\alpha} \otimes \mathcal{O}/(p^r)}} \right| = \left| \frac{V_{\mathbb{Z}}^{\alpha} \otimes \mathbb{Z}/(p^r)}{\ker T|_{V_{\mathbb{Z}}^{\alpha} \otimes \mathbb{Z}/(p^r)}} \right|^{ef}.$$

Taking into account that $(p)^r = \mathfrak{p}^{er}$, where $er > e\alpha$, and applying Lemma (6.2.1) to the left hand side of the above equality we obtain the claim of the Proposition.

(6.3) Slope subspaces of cohomology of irreducible representations. We turn to the situation of interest to us. We fix a reductive, $\mathbb{Q}$-split algebraic group $\tilde{\mathbf{G}}$, and we use the notations introduced in section (5.1). Thus, $\tilde{\mathbf{G}}(\bar{\mathbb{Q}}) = \mathbf{G}(\bar{\mathbb{Q}}) \times_{\mu} \mathbb{G}_m^a(\bar{\mathbb{Q}})$, where $\mathbf{G} = \mathbf{G}_{\lambda_0}$ is the derived group of $\tilde{\mathbf{G}}$ and $\mu \leq \tilde{\mathbf{G}}(\bar{\mathbb{Q}}) \cap \mathbb{G}_m^a(\bar{\mathbb{Q}})$ is a finite subgroup; moreover, $\tilde{\mathbf{T}}$ is a maximal split torus in $\tilde{\mathbf{G}}$ as defined in section (5.1), i.e. $\tilde{\mathbf{T}}(\bar{\mathbb{Q}}) = \mathbf{T}(\bar{\mathbb{Q}}) \times_{\mu} \mathbb{G}_m^a(\bar{\mathbb{Q}})$, where $\mathbf{T}$ is a maximal split torus in $\mathbf{G}$ as defined in (4.3). We fix the natural $\mathbb{Z}$-structure on $\mathbf{G}_{\lambda_0}$ and we select an arithmetic subgroup Γ of $\mathbf{G}(\mathbb{Z})$, which satisfies the following local condition at p:

$$\Gamma \leq K_*(p).$$

(cf. section (4.2) for the definition of $K_*(p)$). Moreover, $\tilde{\lambda} \in X(\tilde{\mathbf{T}})$ denotes any dominant and integral weight as in equation (3) in section (5.1). As in section (5.2) we fix strictly positive integers $e_\alpha \in \mathbb{N}$ and an element $h \in \tilde{\mathbf{T}}(\mathbb{Q})$ satisfying $v_p(\alpha^\circ(h)) = e_\alpha$ for all $\alpha \in \Delta$ and we define the normalized Hecke operator

$$\mathbb{T} = \mathbb{T}(h) = \tilde{\lambda}(h)T(h).$$

We recall that $\mathbb{T}$ defines an operator on $H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}))$ (cf. (5.4) Remark). We denote by $H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}))_{\mathrm{int}}$ the image of the canonical map

$$H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z})) \to H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Q})).$$

Since the embedding is $\mathbb{T}$-equivariant, $H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}))_{\mathrm{int}}$ is a $\mathbb{T}$-stable lattice in $H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Q}))$. Thus, we are in the situation of section (6.1) and we can define the following slope subspaces: $H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Q}))^\alpha$ is the slope α subspace of $H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Q}))$ with respect to $\mathbb{T}$ and we set

$$H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}))^\alpha_{\mathrm{int}} = H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Q}))^\alpha \cap H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}))_{\mathrm{int}}.$$

Since $H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Q}))^\alpha$ is $\mathbb{T}$ stable we deduce that $H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}))^\alpha_{\mathrm{int}}$ is a $\mathbb{T}$-stable lattice in $H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Q}))^\alpha$.

(6.4) Proposition. *There is an exact sequence of $\mathbb{T} \times \mathbb{Z}/(p^r)$-modules*

$$\mathcal{X} \xrightarrow{j} H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}))_{\mathrm{int}} \otimes \mathbb{Z}/(p^r) \to H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z})/L_{\tilde{\lambda}}(\mathbb{Z}, r))/\mathcal{Y},$$

where $\mathcal{Y} \leq H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z})/L_{\tilde{\lambda}}(r, \mathbb{Z}))$ is a submodule and $\mathcal{X}$ is annihilated by $\mathbb{T}$ (i.e. $j(\mathcal{X})$ is contained in the kernel of $\mathbb{T}$).

Proof. For shortness, we set $H^i(V) = H^i(\Gamma, V)$ for any Γ-module V. We start from the short exact sequence

$$0 \to L_{\tilde{\lambda}}(\frac{\mathbb{Z}}{p^r}, r) \xrightarrow{i} L_{\tilde{\lambda}}(\mathbb{Z}) \otimes \mathbb{Z}/(p^r) \xrightarrow{\pi} \frac{L_{\tilde{\lambda}}(\mathbb{Z})}{L_{\tilde{\lambda}}(\mathbb{Z}, r)} \to 0, \tag{5}$$

where i is the inclusion (cf. equation (14) in section (5.5)) and π is defined by $\pi(v \otimes 1) = v + L_{\tilde{\lambda}}(\mathbb{Z}, r)$. From equation (5) we obtain a long exact cohomology sequence

$$\ldots \to H^i(L_{\tilde{\lambda}}(\frac{\mathbb{Z}}{p^r}, r)) \xrightarrow{H(i)} H^i(L_{\tilde{\lambda}}(\mathbb{Z}) \otimes \mathbb{Z}/(p^r)) \xrightarrow{H(\pi)} H^i(\frac{L_{\tilde{\lambda}}(\mathbb{Z})}{L_{\tilde{\lambda}}(\mathbb{Z}, r)}) \to \cdots. \tag{6}$$

We note that equation (6) is a sequence of $\mathbb{T} \times \mathbb{Z}/(p^r)$-modules (cf. (5.4) Remark and (5.6) Remark). The short exact sequence

$$0 \to L_{\tilde{\lambda}}(\mathbb{Z}) \xrightarrow{\cdot p^r} L_{\tilde{\lambda}}(\mathbb{Z}) \xrightarrow{\mathrm{pr}} L_{\tilde{\lambda}}(\mathbb{Z})/p^r L_{\tilde{\lambda}}(\mathbb{Z}) \to 0,$$

where pr is the natural projection, yields a long exact sequence

$$\cdots \to H^i(L_{\tilde{\lambda}}(\mathbb{Z})) \xrightarrow{\cdot p^r} H^i(L_{\tilde{\lambda}}(\mathbb{Z})) \xrightarrow{H(\mathrm{pr})} H^i(L_{\tilde{\lambda}}(\mathbb{Z})/p^r L_{\tilde{\lambda}}(\mathbb{Z})) \to \cdots$$

hence, we obtain an embedding

$$H^i(L_{\tilde\lambda}(\mathbb{Z}))\otimes\mathbb{Z}/(p^r)\overset{H(\mathrm{pr})}{\hookrightarrow}H^i(L_{\tilde\lambda}(\mathbb{Z})\otimes\mathbb{Z}/(p^r)). \tag{7}$$

We identify $H^i(L_{\tilde\lambda}(\mathbb{Z}))\otimes\mathbb{Z}/(p^r)$ with its image under $H(\mathrm{pr})$. Restricting the morphism $H(\pi)$ to $H^i(L_{\tilde\lambda}(\mathbb{Z}))\otimes\mathbb{Z}/(p^r)$ we obtain from equation (6) the exact sequence

$$X\overset{H(i)}{\to}H^i(L_{\tilde\lambda}(\mathbb{Z}))\otimes\mathbb{Z}/(p^r)\overset{H(\pi)}{\to}H^i(\frac{L_{\tilde\lambda}(\mathbb{Z})}{L_{\tilde\lambda}(\mathbb{Z},r)}), \tag{8}$$

where

$$X=H^i(L_{\tilde\lambda}(\frac{\mathbb{Z}}{p^r},r))\cap H(i)^{-1}\big(H^i(L_{\tilde\lambda}(\mathbb{Z}))\otimes\mathbb{Z}/(p^r)\big).$$

We denote by $\mathcal{T}_\lambda$ the torsion submodule of $H^i(L_{\tilde\lambda}(\mathbb{Z}))$ and we set $\mathcal{T}_{\lambda,r}=\mathcal{T}_\lambda\otimes\mathbb{Z}/(p^r)$. From equation (8) we further obtain the exact sequence

$$\frac{X}{H(i)^{-1}(\mathcal{T}_{\lambda,r})}\overset{H(i)}{\to}\frac{H^i(L_{\tilde\lambda}(\mathbb{Z}))\otimes\mathbb{Z}/(p^r)}{\mathcal{T}_{\lambda,r}}\overset{H(\pi)}{\to}\frac{H^i(L_{\tilde\lambda}(\mathbb{Z})/L_{\tilde\lambda}(\mathbb{Z},r))}{H^i(\pi)(\mathcal{T}_{\lambda,r})}. \tag{9}$$

We note that equation (9) is a sequence of $\mathbb{T}\times\mathbb{Z}/(p^r)$-modules, because $\mathbb{T}$ leaves the torsion $\mathcal{T}$ invariant. We set $\mathcal{X}=X/H(i)^{-1}(\mathcal{T}_{\lambda,r})$. Since $\mathbb{T}$ annihilates $H^i(L_{\tilde\lambda}(\frac{\mathbb{Z}}{p^r},r))$ (cf. (5.5) Lemma) we see that $\mathbb{T}$ annihilates $\mathcal{X}$. Since moreover,

$$\frac{H^i(L_{\tilde\lambda}(\mathbb{Z}))\otimes\mathbb{Z}/(p^r)}{\mathcal{T}_{\lambda,r}}\cong H^i(L_{\tilde\lambda}(\mathbb{Z}))_{\mathrm{int}}\otimes\mathbb{Z}/(p^r)$$

we see that equation (9) yields the claim. Thus, the Proposition is proven.

(6.5) Boundedness of the slope subspaces. We keep the notations introduced in section (6.3) and (6.4). In addition we set

$$d(\tilde\lambda,i,\alpha)=\dim_{\mathbb{Z}}H^i(\Gamma,L_{\tilde\lambda}(\mathbb{Z}))^\alpha_{\mathrm{int}}=\dim_{\mathbb{Q}}H^i(\Gamma,L_{\tilde\lambda}(\mathbb{Q}))^\alpha.$$

Theorem. *Let $\beta\in\mathbb{Q}_{\geq0}$ and choose $i\in\mathbb{N}_0$. There is $C=C_{\beta,i}\in\mathbb{N}$, not depending on the weight $\tilde\lambda$, such that for all integral and dominant weights $\tilde\lambda$ we have*

$$\sum_{0\leq\alpha\leq\beta}\dim H^i(\Gamma,L_{\tilde\lambda}(\mathbb{Q}))^\alpha\leq C.$$

Proof. We set $r=[\beta]+1$, hence, r is the smallest integer such that $r>\beta$. Replacing $\mathcal{X}$ in (6.4) Proposition by its image under j we may assume that $\mathcal{X}$ is a submodule of $H^i(\Gamma,L_{\tilde\lambda}(\mathbb{Z}))_{\mathrm{int}}\otimes\mathbb{Z}/(p^r)$. We denote by $\pi_\alpha:\bigoplus_{\gamma\geq0}H^i(L_{\tilde\lambda}(\mathbb{Z}))^\gamma_{\mathrm{int}}\otimes\mathbb{Z}/(p^r)\to H^i(L_{\tilde\lambda}(\mathbb{Z}))^\alpha_{\mathrm{int}}\otimes\mathbb{Z}/(p^r)$ the canonical projection and we set

$$\mathcal{X}'=\mathcal{X}\cap\Big(\bigoplus_{\alpha\geq0}H^i(L_{\tilde\lambda}(\mathbb{Z}))^\alpha_{\mathrm{int}}\otimes\mathbb{Z}/(p^r)\Big).$$

There is a surjective map

$$\frac{\bigoplus_{\alpha\geq 0} H^i(L_{\tilde\lambda}(\mathbb{Z}))^\alpha_{\text{int}}\otimes\mathbb{Z}/(p^r)}{\mathcal{X}'} \to \bigoplus_{\alpha\geq 0}\frac{H^i(L_{\tilde\lambda}(\mathbb{Z}))^\alpha_{\text{int}}\otimes\mathbb{Z}/(p^r)}{\pi_\alpha(\mathcal{X}')}. \tag{10}$$

sending $(\sum_{\alpha\geq 0} v_\alpha)+\mathcal{X}'$ to $\sum_{\alpha\geq 0} v_\alpha+\pi_\alpha(\mathcal{X}')$. Let $v=\sum_{\alpha\geq 0}\pi_\alpha(v)\in\mathcal{X}'$ be arbitrary. Since $\mathcal{X}'\subseteq\mathcal{X}$ is contained in the kernel of $\mathbb{T}$ (cf. (6.4) Proposition) we know that $0=\mathbb{T}(v)=\sum_{\alpha\geq 0}\mathbb{T}\pi_\alpha(v)$, hence, $\mathbb{T}\pi_\alpha(v)=0$ for all $\alpha\geq 0$. Thus, we obtain

$$\pi_\alpha(\mathcal{X}')\leq \ker\mathbb{T}|_{H^i(L_{\tilde\lambda}(\mathbb{Z}))^\alpha_{\text{int}}\otimes\mathbb{Z}/(p^r)}. \tag{11}$$

Since $\bigoplus_{\alpha\geq 0} H^i(L_{\tilde\lambda}(\mathbb{Z}))^\alpha_{\text{int}}\otimes\mathbb{Z}/(p^r)\leq H^i(L_{\tilde\lambda}(\mathbb{Z}))_{\text{int}}\otimes\mathbb{Z}/(p^r)$ we further obtain

$$\begin{aligned}
\left|\bigoplus_{\alpha\geq 0}\frac{H^i(L_{\tilde\lambda}(\mathbb{Z}))^\alpha_{\text{int}}\otimes\mathbb{Z}/(p^r)}{\ker\mathbb{T}|_{H^i(L_{\tilde\lambda}(\mathbb{Z}))^\alpha_{\text{int}}\otimes\mathbb{Z}/(p^r)}}\right| &\leq \left|\bigoplus_{\alpha\geq 0}\frac{H^i(L_{\tilde\lambda}(\mathbb{Z}))^\alpha_{\text{int}}\otimes\mathbb{Z}/(p^r)}{\pi_\alpha(\mathcal{X}')}\right| \quad \text{(eq. (11))}\\
&\leq \left|\frac{\bigoplus_{\alpha\geq 0} H^i(L_{\tilde\lambda}(\mathbb{Z}))^\alpha_{\text{int}}\otimes\mathbb{Z}/(p^r)}{\mathcal{X}'}\right| \quad \text{(eq. (10))}\\
&\leq \left|\frac{H^i(L_{\tilde\lambda}(\mathbb{Z}))_{\text{int}}\otimes\mathbb{Z}/(p^r)}{\mathcal{X}}\right|.
\end{aligned}$$

Thus, the exactness of the sequence in (6.4) Proposition yields

$$\left|\bigoplus_{\alpha\geq 0}\frac{H^i(L_{\tilde\lambda}(\mathbb{Z}))^\alpha_{\text{int}}\otimes\mathbb{Z}/(p^r)}{\ker\mathbb{T}|_{H^i(L_{\tilde\lambda}(\mathbb{Z}))^\alpha_{\text{int}}\otimes\mathbb{Z}/(p^r)}}\right| \leq \left|H^i(L_{\tilde\lambda}(\mathbb{Z})/L_{\tilde\lambda}(\mathbb{Z},r))\right|.$$

On the other hand, since $\beta<r$, (6.2.2) Proposition implies

$$\prod_{0\leq\alpha\leq\beta} p^{d(\tilde\lambda,i,\alpha)(r-\alpha)}\left|\;\left|\bigoplus_{0\leq\alpha\leq\beta}\frac{H^i(L_{\tilde\lambda}(\mathbb{Z}))^\alpha_{\text{int}}\otimes\mathbb{Z}/(p^r)}{\ker\mathbb{T}|_{H^i(L_{\tilde\lambda}(\mathbb{Z}))^\alpha_{\text{int}}\otimes\mathbb{Z}/(p^r)}}\right|\right.. \tag{12}$$

Hence, we obtain

$$\prod_{0\leq\alpha\leq\beta} p^{d(\tilde\lambda,i,\alpha)(r-\alpha)}\leq |H^i(\Gamma,L_{\tilde\lambda}(\mathbb{Z})/L_{\tilde\lambda}(\mathbb{Z},r))| = |H^i(\Gamma,L_\lambda(\mathbb{Z})/L_\lambda(\mathbb{Z},r))|.$$

Now, the Proposition in section (4.4.2) implies that $L_\lambda(\mathbb{Z})/L_\lambda(\mathbb{Z},r))$ and, hence, $|H^i(\Gamma,L_\lambda(\mathbb{Z})/L_\lambda(\mathbb{Z},r))|$ only depends on the coset $\lambda+p^{2r}\Gamma_{\text{ad}}$ if $\langle\lambda,\alpha\rangle\geq r$ for all simple roots α. Since $\Gamma_{\text{sc}}/p^{2r}\Gamma_{\text{ad}}$ as well as the set $\mathcal{B}$ of all dominant and integral weights λ satisfying $\langle\lambda,\alpha\rangle\geq r$ for all simple roots α are finite sets, we finally obtain:

$$\sum_{0\leq\alpha\leq\beta} d(\tilde\lambda,i,\alpha)(r-\alpha)\leq\log_p M,$$

where
$$M = \max_\lambda |H^i(\Gamma, L_\lambda(\mathbb{Z})/L_\lambda(\mathbb{Z}, r))|,$$
and λ runs over the union of $\mathcal{B}$ with a set of representatives for $\Gamma_{\mathrm{sc}}/p^{2r}\Gamma_{\mathrm{ad}}$. This completes the proof of the Theorem.

(6.6) $\mathbb{Q}_p$-split groups. We assume weaker that the reductive group $\tilde{\mathbf{G}}$ is defined over $\mathbb{Q}$ and split over $\mathbb{Q}_p$. Hence, $\tilde{\mathbf{G}} \cong \mathbf{G} \times_\mu \mathbb{G}_m^a$ over $\mathbb{Q}_p$, where

$$\mathbf{G} \cong \mathbf{G}_{\lambda_0}/\mathbb{Q}_p \tag{13}$$

for some irreducible representation $(\rho_{\lambda_0}, L_{\lambda_0})$ of the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$. Equation (13) in particular defines a $\mathbb{Z}_p$-structure on $\mathbf{G}$. We fix a prime $p \in \mathbb{N}$ and we let $\Gamma \leq K_*(p)$ be a subgroup. Moreover, we choose strictly positive integers e_α, $\alpha \in \Delta$, and an element $h \in \tilde{\mathbf{T}}(\mathbb{Q}_p)$ such that $v_p(\alpha^\circ(h)) = e_\alpha$ for all $\alpha \in \Delta$. As in (5.2) this yields a normalized Hecke operator
$$\mathbb{T} = p^{v_p(\tilde{\lambda}(h))}T \qquad (T = \Gamma h \Gamma),$$
which now acts on cohomology with *p-adic* integral coefficients (compare (5.4) Remark, (5.6) Remark), e.g. $\mathbb{T}$ acts on $H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Z}_p))$. Following the proof of (6.5) Theorem we then obtain the following p-adic version of (6.5) Theorem.

Theorem. *Let $\beta \in \mathbb{Q}_{\geq 0}$ and choose $i \in \mathbb{N}_0$. There is $C = C_{\beta,i} \in \mathbb{N}$ such that for all integral and dominant weights $\tilde{\lambda}$ we have*
$$\sum_{0 \leq \alpha \leq \beta} \dim H^i(\Gamma, L_{\tilde{\lambda}}(\mathbb{Q}_p))^\alpha \leq C.$$

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